

8

Alternating-current circuits

Overview In earlier chapters we encountered resistors, capacitors, and inductors. We will now study circuits containing all three of these elements. If such a circuit contains no emf source, the current takes the form of a *decaying oscillation* (in the case of small damping). The rate of decay is described by the *Q factor*. If we add on a sinusoidally oscillating emf source, then the current will reach a *steady state* with the same frequency of oscillation as the emf source. However, in general there will be a *phase difference* between the current and the emf. This phase, along with the amplitude of the current, can be determined by three methods. The first method is to guess a *sinusoidal* solution to the differential equation representing the Kirchhoff loop equation. The second is to guess a *complex exponential* solution and then take the real part to obtain the actual current. The third is to use complex voltages, currents, and impedances. These *complex impedances* can be combined via the same series and parallel rules that work for resistors. As we will see, the third method is essentially the same as the second method, but with better bookkeeping; this makes it far more tractable in the case of complicated circuits. Finally, we derive an expression for the power dissipated in a circuit, which reduces to the familiar V^2/R result if the circuit is purely resistive.

8.1 A resonant circuit

A mass attached to a spring is a familiar example of an oscillator. If the amplitude of oscillation is not too large, the motion will be a sinusoidal function of the time. In that case, we call it a *harmonic oscillator*.

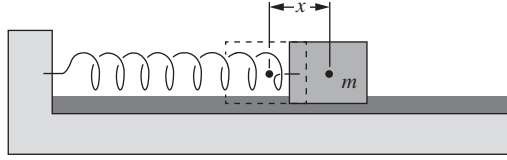


Figure 8.1.
A mechanical damped harmonic oscillator.

The characteristic feature of any mechanical harmonic oscillator is a restoring force proportional to the displacement of a mass m from its position of equilibrium, $F = -kx$ (Fig. 8.1). In the absence of other external forces, the mass, if initially displaced, will oscillate with unchanging amplitude at the angular frequency $\omega = \sqrt{k/m}$. But usually some kind of friction will bring it eventually to rest. The simplest case is that of a retarding force proportional to the velocity of the mass, dx/dt . Motion in a viscous fluid provides an example. A system in which the restoring force is proportional to some displacement x and the retarding force is proportional to the time derivative dx/dt is called a *damped harmonic oscillator*.

An electric circuit containing capacitance and inductance has the essentials of a harmonic oscillator. Ohmic resistance makes it a damped harmonic oscillator. Indeed, thanks to the extraordinary linearity of actual electric circuit elements, the electrical damped harmonic oscillator is more nearly ideal than most mechanical oscillators. The system we shall study first is the “series RLC ” circuit shown in Fig. 8.2. Note that there is no emf in this circuit. We will introduce an $\mathcal{E}$ (an oscillating one) in Section 8.2.

Let Q be the charge, at time t , on the capacitor in this circuit. The potential difference, or voltage across the capacitor, is V , which obviously is the same as the voltage across the series combination of inductor L and resistor R . We take V to be positive when the upper capacitor plate is positively charged, and we define the positive current direction by the arrow in Fig. 8.2. With the signs chosen that way, the relations connecting charge Q , current I , and voltage across the capacitor V are

$$I = -\frac{dQ}{dt}, \quad Q = CV, \quad V = L\frac{dI}{dt} + RI. \quad (8.1)$$

We want to eliminate two of the three variables Q , I , and V . Let us write Q and I in terms of V . From the first two equations we obtain $I = -C dV/dt$, and the third equation becomes $V = -LC(d^2V/dt^2) - RC(dV/dt)$, or

$$\frac{d^2V}{dt^2} + \left(\frac{R}{L}\right)\frac{dV}{dt} + \left(\frac{1}{LC}\right)V = 0. \quad (8.2)$$

This equation takes exactly the same form as the $F = ma$ equation for a mass on the end of a spring immersed in a fluid in which the damping force is $-bv$, where b is the damping coefficient and v is the velocity.

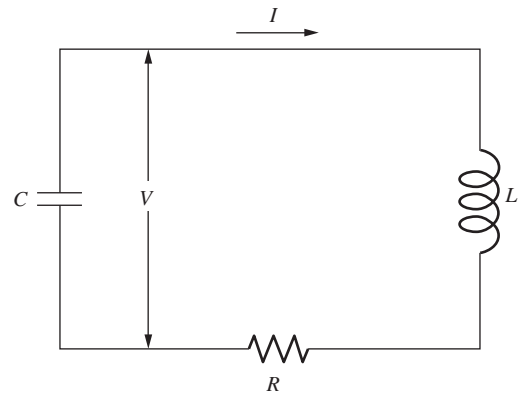


Figure 8.2.
A “series RLC ” circuit.

The $F = ma$ equation for that system is $-kx - b\dot{x} = m\ddot{x}$. We can compare this with Eq. (8.2) (after multiplying through by L):

$$L \frac{d^2V}{dt^2} + R \frac{dV}{dt} + \left(\frac{1}{C}\right)V = 0 \iff m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0. \quad (8.3)$$

We see that the inductance L is the analog of the mass m ; this element provides the inertia that resists change. The resistance R is the analog of the damping coefficient b ; this element causes energy dissipation. And the inverse of the capacitance, $1/C$, is the analog of the spring constant k ; this element provides the restoring force. (There isn't anything too deep about the reciprocal form of $1/C$ here; we could have just as easily defined a quantity $C' \equiv 1/C$, with $V = C'Q$.)

Equation (8.2) is a second-order differential equation with constant coefficients. We shall try a solution of the form

$$V(t) = Ae^{-\alpha t} \cos \omega t, \quad (8.4)$$

where A , α , and ω are constants. (See Problem 8.3 for an explanation of where this form comes from.) The first and second derivatives of this function are

$$\begin{aligned} \frac{dV}{dt} &= Ae^{-\alpha t} [-\alpha \cos \omega t - \omega \sin \omega t], \\ \frac{d^2V}{dt^2} &= Ae^{-\alpha t} [(\alpha^2 - \omega^2) \cos \omega t + 2\alpha\omega \sin \omega t]. \end{aligned} \quad (8.5)$$

Substituting back into Eq. (8.2), we cancel out the common factor $Ae^{-\alpha t}$ and are left with

$$\begin{aligned} &(\alpha^2 - \omega^2) \cos \omega t + 2\alpha\omega \sin \omega t - \frac{R}{L}(\alpha \cos \omega t + \omega \sin \omega t) \\ &+ \frac{1}{LC} \cos \omega t = 0. \end{aligned} \quad (8.6)$$

This will be satisfied for all t if, and only if, the coefficients of $\sin \omega t$ and $\cos \omega t$ are both zero. That is, we must require

$$2\alpha\omega - \frac{R\omega}{L} = 0 \quad \text{and} \quad \alpha^2 - \omega^2 - \alpha \frac{R}{L} + \frac{1}{LC} = 0. \quad (8.7)$$

The first of these equations gives a condition on α :

$$\boxed{\alpha = \frac{R}{2L}} \quad (8.8)$$

while the second equation requires that

$$\omega^2 = \frac{1}{LC} - \alpha \frac{R}{L} + \alpha^2 \implies \boxed{\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2}} \quad (8.9)$$

We are assuming that the ω in Eq. (8.4) is a real number, so ω^2 cannot be negative. Therefore we succeed in obtaining a solution of the form assumed in Eq. (8.4) only if $R^2/4L^2 \leq 1/LC$. In fact, it is the case of “light damping,” that is, low resistance, that we want to examine, so we shall assume that the values of R , L , and C in the circuit are such that the inequality $R < 2\sqrt{L/C}$ holds. However, see the end of this section for a brief discussion of the $R = 2\sqrt{L/C}$ and $R > 2\sqrt{L/C}$ cases.

The function $Ae^{-\alpha t} \cos \omega t$ is not the only possible solution; $Be^{-\alpha t} \sin \omega t$ works just as well, with the same requirements, Eqs. (8.8) and (8.9), on α and ω , respectively. The general solution is the sum of these:

$$V(t) = e^{-\alpha t} (A \cos \omega t + B \sin \omega t) \quad (8.10)$$

The arbitrary constants A and B could be adjusted to fit initial conditions. That is not very interesting. Whether the solution in any given case involves the sine or the cosine function, or some superposition, is a trivial matter of how the clock is set. The essential phenomenon is a damped sinusoidal oscillation.

The variation of voltage with time is shown in Fig. 8.3(a). Of course, this cannot really hold for all *past* time. At some time in the past the circuit must have been provided with energy somehow, and then left running. For instance, the capacitor might have been charged, with the circuit open, and then connected to the coil.

In Fig. 8.3(b) the time scale has been expanded, and the dashed curve showing the variation of the current I has been added. For V let us take the damped cosine, Eq. (8.4). Then the current as a function of time is given by

$$I(t) = -C \frac{dV}{dt} = AC\omega \left(\sin \omega t + \frac{\alpha}{\omega} \cos \omega t \right) e^{-\alpha t}. \quad (8.11)$$

The ratio α/ω is a measure of the damping. This is true because if α/ω is very small, many oscillations occur while the amplitude is decaying only a little. For Fig. 8.3 we chose a case in which $\alpha/\omega \approx 0.04$. Then the cosine term in Eq. (8.11) doesn't amount to much. All it does, in effect, is shift the phase by a small angle, $\tan^{-1}(\alpha/\omega)$. So the current oscillation is almost exactly one-quarter cycle out of phase with the voltage oscillation.

The oscillation involves a transfer of energy back and forth from the capacitor to the inductor, or from electric field to magnetic field. At the times marked 1 in Fig. 8.3(b) all the energy is in the electric field. A quarter-cycle later, at 2, the capacitor is discharged and nearly all this energy is found in the magnetic field of the coil. Meanwhile, the circuit resistance R is taking its toll, and as the oscillation goes on, the energy remaining in the fields gradually diminishes.

The relative damping in an oscillator is often expressed by giving a number called Q . This number Q (not to be confused with the charge on the capacitor!) is said to stand for *quality* or *quality factor*. In fact, no

one calls it that; we just call it Q . The less the damping, the larger the number Q . For an oscillator with frequency ω , Q is the dimensionless ratio formed as follows:

$$Q = \omega \cdot \frac{\text{energy stored}}{\text{average power dissipated}} \quad (8.12)$$

Or you may prefer to remember Q as follows:

- Q is the number of radians of the argument ωt (that is, 2π times the number of cycles) required for the energy in the oscillator to diminish by the factor $1/e$.

In our circuit the stored energy is proportional to V^2 or I^2 and, therefore, to $e^{-2\alpha t}$. So the energy decays by $1/e$ in a time $t = 1/2\alpha$, which covers $\omega t = \omega/2\alpha$ radians. Hence, for our RLC circuit, using Eq. (8.8),

$$Q = \frac{\omega}{2\alpha} = \frac{\omega L}{R}. \quad (8.13)$$

You should verify that Eq. (8.12) gives the same result.

What is Q for the oscillation represented in Fig. 8.3? The energy decreases by a factor $1/e$ when V decreases by a factor $1/\sqrt{e} \approx 0.6$. As a rough estimate, this decrease occurs after about two oscillations, which is roughly 13 radians. So $Q \approx 13$.

A special case of the above circuit is where $R = 0$. In this case we have the completely undamped oscillator, whose frequency ω_0 is given by Eq. (8.9) as

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad (8.14)$$

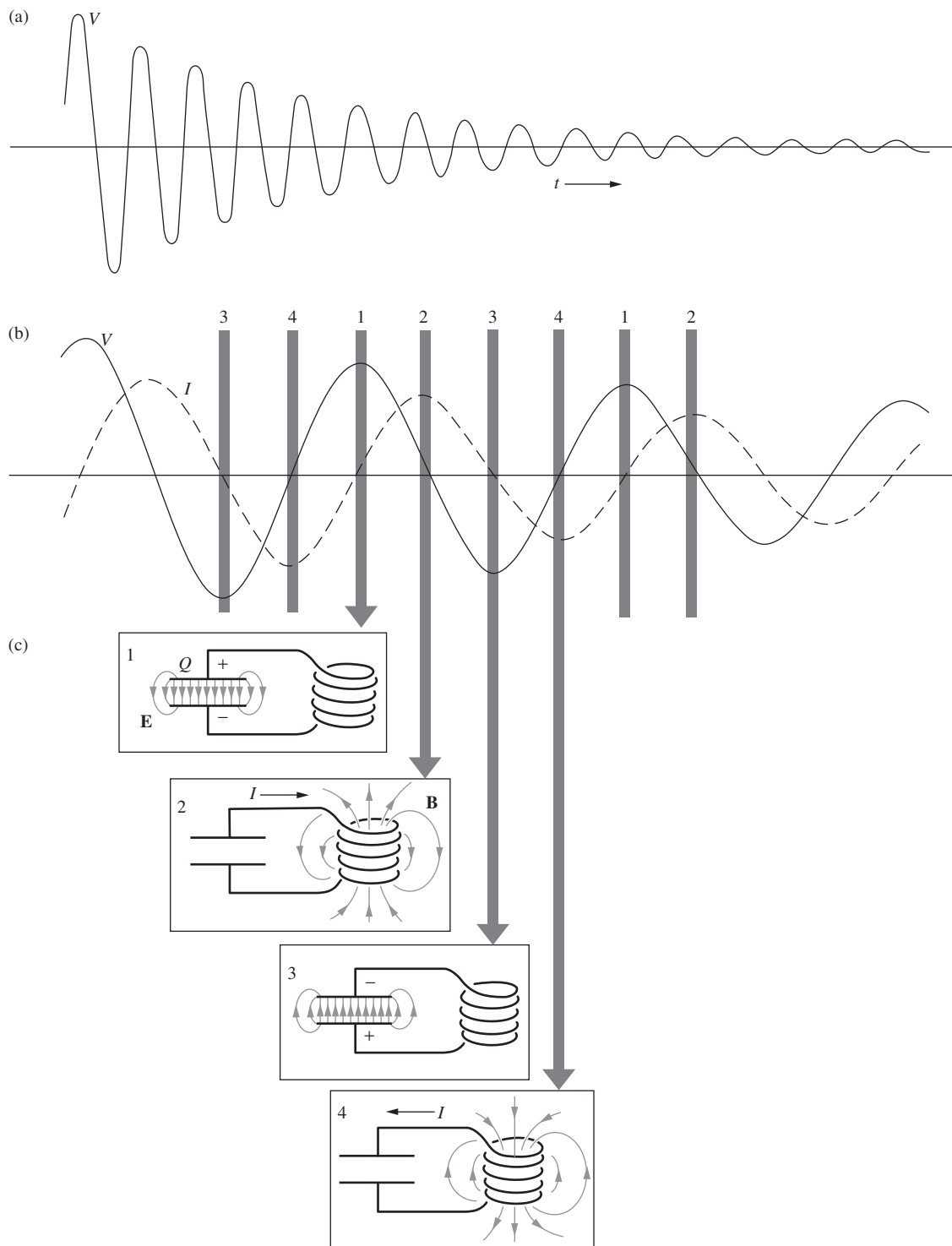
Mostly we deal with systems in which the damping is small enough to be ignored in calculating the frequency. As we can see from Eq. (8.9), and as Problem 8.5 and Exercise 8.18 will demonstrate, light damping has only a second-order effect on ω . Note that in view of Eq. (8.3), the $1/\sqrt{LC}$ frequency for our undamped resonant circuit is the analog of the familiar $\sqrt{k/m}$ frequency for an undamped mechanical oscillator.

For completeness we review briefly what goes on in the overdamped circuit, in which $R > 2\sqrt{L/C}$. Equation (8.2) then has a solution of the form $V = Ae^{-\beta t}$ for two values of β , the general solution being

$$V(t) = Ae^{-\beta_1 t} + Be^{-\beta_2 t}. \quad (8.15)$$

Figure 8.3.

(a) The damped sinusoidal oscillation of voltage in the RLC circuit. (b) A portion of (a) with the time scale expanded and the graph of the current I included. (c) The periodic transfer of energy from electric field to magnetic field and back again. Each picture represents the condition at times marked by the corresponding number in (b).



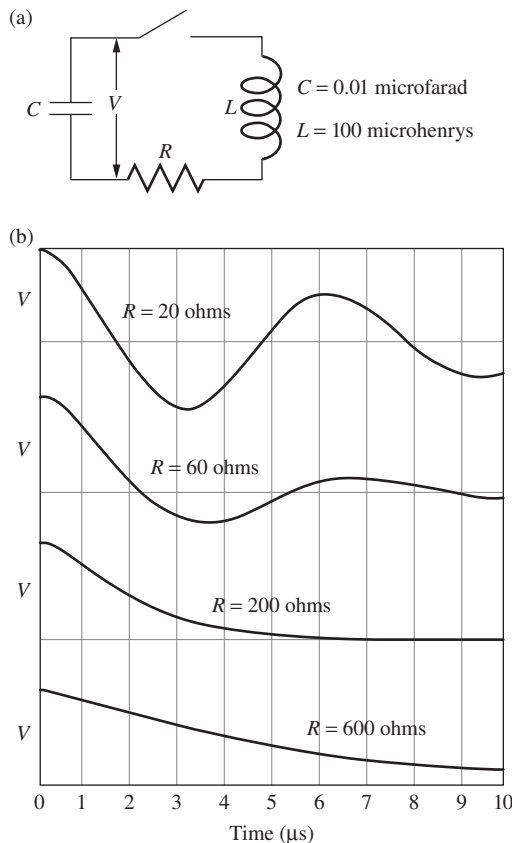


Figure 8.4.

(a) With the capacitor charged, the switch is closed at $t = 0$. (b) Four cases are shown, one of which, $R = 200$ ohms, is the case of critical damping.

There are no oscillations, only a monotonic decay (after perhaps one local extremum, depending on the initial conditions). The task of Problem 8.4 is to find the values of β_1 and β_2 .

In the special case of “critical” damping, where $R = 2\sqrt{L/C}$, we have $\beta_1 = \beta_2$. It turns out (see Problem 8.2) that in this case the solution of the differential equation, Eq. (8.2), takes the form,

$$V(t) = (A + Bt)e^{-\beta t}. \quad (8.16)$$

This is the condition, for given L and C , in which the total energy in the circuit is most rapidly dissipated; see Exercise 8.23.

You can see this whole range of behavior in Fig. 8.4, where $V(t)$ is plotted for two underdamped circuits, a critically damped circuit, and an overdamped circuit. The capacitor and inductor remain the same; only the resistor is changed. The natural angular frequency $\omega_0 = 1/\sqrt{LC}$ is 10^6 s^{-1} for this circuit, corresponding to a frequency in cycles per second of $10^6/2\pi$, or 159 kilocycles per second.

The circuit is started off by charging the capacitor to a potential difference of, say, 1 volt and then closing the switch at $t = 0$. That is, $V = 1$ at $t = 0$ is one initial condition. Also, $I = 0$ at $t = 0$, because the inductor will not allow the current to rise discontinuously. Therefore, the other initial condition on V is $dV/dt = 0$, at $t = 0$. Note that all four decay curves start the same way. In the heavily damped case ($R = 600$ ohms) most of the decay curve looks like the simple exponential decay of an RC circuit. Only the very beginning, where the curve is rounded over so that it starts with zero slope, betrays the presence of the inductance L .

8.2 Alternating current

The resonant circuit we have just discussed contained no source of energy and was, therefore, doomed to a *transient* activity, an oscillation that must sooner or later die out (unless $R = 0$ exactly). In an alternating-current circuit we are concerned with a *steady state*, a current and voltage oscillating sinusoidally without change in amplitude. Some oscillating electromotive force drives the system.

The frequency f of an alternating current is ordinarily expressed in cycles per second (or Hertz (Hz), after the discoverer¹ of electromagnetic waves). The angular frequency $\omega = 2\pi f$ is the quantity that usually appears in our equations. It will always be assumed to be in radians/second. That unit has no special name; we write it simply s^{-1} . Thus our familiar (in North America) 60 Hz current has $\omega = 377 \text{ s}^{-1}$. But, in general, ω can take on any value we choose; it need not have anything to do with the frequency ω we found in the previous section in Eq. (8.9).

¹ In 1887, at the University of Karlsruhe, Heinrich Hertz demonstrated electromagnetic waves generated by oscillating currents in a macroscopic electric circuit. The frequencies were around 10^9 cycles per second, corresponding to wavelengths around 30 cm. Although Maxwell’s theory, developed 15 years earlier, had left little doubt that light must be an electromagnetic phenomenon, in the history of electromagnetism Hertz’s experiments were an immensely significant turning point.

Our goal in this section is to determine how the current behaves in a series RLC circuit with an oscillating voltage source. To warm up, we consider a few simpler circuits first. In [Section 8.3](#) we provide an alternative method for solving the RLC circuit. This method uses complex exponentials in a rather slick way. In [Sections 8.4](#) and [8.5](#) we generalize this complex-exponential method in a manner that allows us to treat an alternating-current circuit (involving resistors, inductors, and capacitors) in essentially the same simple way that we treat a direct-current circuit involving only resistors.

8.2.1 RL circuit

Let us apply an electromotive force $\mathcal{E} = \mathcal{E}_0 \cos \omega t$ to a circuit containing inductance and resistance. We might generate $\mathcal{E}$ by a machine schematically like the one in [Fig. 7.13](#), having provided some engine or motor to turn the shaft at the constant angular speed ω . The symbol at the left in [Fig. 8.5](#) is a conventional way to show the presence of an alternating electromotive force in a circuit. It suggests a generator connected in series with the rest of the circuit. But you need not think of an electromotive force as located at a particular place in the circuit. It is only the line integral around the whole circuit that matters. [Figure 8.5](#) could just as well represent a circuit in which the electromotive force arises from a changing magnetic field over the whole area enclosed by the circuit.

We set the sum of voltage drops over the elements of this circuit equal to the electromotive force $\mathcal{E}$, exactly as we did in developing [Eq. \(7.66\)](#). The equation governing the current is then

$$L \frac{dI}{dt} + RI = \mathcal{E}_0 \cos \omega t. \quad (8.17)$$

There may be some transient behavior, depending on the initial conditions, that is, on how and when the generator is switched on. But we are interested only in the steady state, when the current is oscillating obediently at the frequency of the driving force, with the amplitude and phase necessary to keep [Eq. \(8.17\)](#) satisfied. To show that this is possible, consider a current described by

$$I(t) = I_0 \cos(\omega t + \phi) \quad (8.18)$$

To determine the constants I_0 and ϕ , we put this into [Eq. \(8.17\)](#):

$$-LI_0\omega \sin(\omega t + \phi) + RI_0 \cos(\omega t + \phi) = \mathcal{E}_0 \cos \omega t. \quad (8.19)$$

The functions $\sin \omega t$ and $\cos \omega t$ can be separated out:

$$\begin{aligned} -LI_0\omega (\sin \omega t \cos \phi + \cos \omega t \sin \phi) \\ + RI_0 (\cos \omega t \cos \phi - \sin \omega t \sin \phi) = \mathcal{E}_0 \cos \omega t. \end{aligned} \quad (8.20)$$

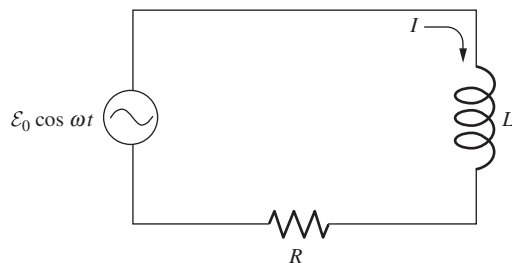
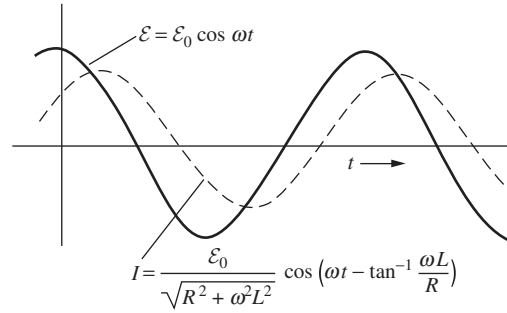


Figure 8.5.

A circuit with inductance, driven by an alternating electromotive force.

**Figure 8.6.**

The current I_1 in the circuit of Fig. 8.5, plotted along with the electromotive force $\mathcal{E}$ on the same time scale. Note the phase difference.

Setting the coefficients of $\sin \omega t$ and $\cos \omega t$ separately equal to zero gives, respectively,

$$-LI_0\omega \cos \phi - RI_0 \sin \phi = 0 \implies \boxed{\tan \phi = -\frac{\omega L}{R}} \quad (8.21)$$

and

$$-LI_0\omega \sin \phi + RI_0 \cos \phi - \mathcal{E}_0 = 0, \quad (8.22)$$

which gives

$$\begin{aligned} I_0 &= \frac{\mathcal{E}_0}{R \cos \phi - \omega L \sin \phi} \\ &= \frac{\mathcal{E}_0}{R(\cos \phi + \tan \phi \sin \phi)} = \frac{\mathcal{E}_0 \cos \phi}{R}. \end{aligned} \quad (8.23)$$

Since Eq. (8.21) implies²

$$\cos \phi = \frac{R}{\sqrt{R^2 + \omega^2 L^2}}, \quad (8.24)$$

we can write I_0 as

$$\boxed{I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + \omega^2 L^2}}} \quad (8.25)$$

In Fig. 8.6 the oscillations of $\mathcal{E}$ and I are plotted on the same graph. Since ϕ is a negative angle, the current reaches its maximum a bit *later* than the electromotive force. One says, “The current lags the voltage in an inductive circuit.” The quantity ωL , which has the dimensions of resistance and can be expressed in ohms, is called the *inductive reactance*.

² The $\tan \phi$ expression in Eq. (8.21) actually gives only the magnitude of $\cos \phi$ and not the sign, since ϕ could lie in the second or fourth quadrants. But since the convention is to take I_0 and $\mathcal{E}_0$ positive, Eq. (8.23) tells us that $\cos \phi$ is positive. The angle ϕ therefore lies in the fourth quadrant, at least for an RL circuit.

8.2.2 RC circuit

If we replace the inductor L by a capacitor C , as in Fig. 8.7, we have a circuit governed by the equation

$$-\frac{Q}{C} + RI = \mathcal{E}_0 \cos \omega t, \quad (8.26)$$

where we have defined Q to be the charge on the bottom plate of the capacitor, as shown. We again consider the steady-state solution

$$I(t) = I_0 \cos(\omega t + \phi). \quad (8.27)$$

Since $I = -dQ/dt$, we have

$$Q = -\int I dt = -\frac{I_0}{\omega} \sin(\omega t + \phi). \quad (8.28)$$

Note that, in going from I to Q by integration, there is no question of adding a constant of integration, for we know that Q must oscillate symmetrically about zero in the steady state. Substituting Q back into Eq. (8.26) leads to

$$\frac{I_0}{\omega C} \sin(\omega t + \phi) + RI_0 \cos(\omega t + \phi) = \mathcal{E}_0 \cos \omega t. \quad (8.29)$$

Just as before, we obtain conditions on ϕ and I_0 by requiring that the coefficients of $\sin \omega t$ and $\cos \omega t$ separately vanish. Alternatively, we can avoid this process by noting that, in going from Eq. (8.19) to Eq. (8.29), we have simply traded $-\omega L$ for $1/\omega C$. The results analogous to Eqs. (8.21) and (8.25) are therefore

$$\boxed{\tan \phi = \frac{1}{R\omega C}} \quad \text{and} \quad \boxed{I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + (1/\omega C)^2}}} \quad (8.30)$$

Note that the phase angle is now positive, that is, it lies in the first quadrant. (The result in Eq. (8.23) is unchanged, so $\cos \phi$ is again positive. But $\tan \phi$ is now also positive.) As the saying goes, the current “leads the voltage” in a capacitive circuit. What this means is apparent in the graph of Fig. 8.8.

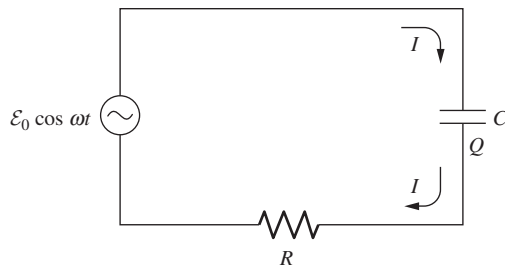
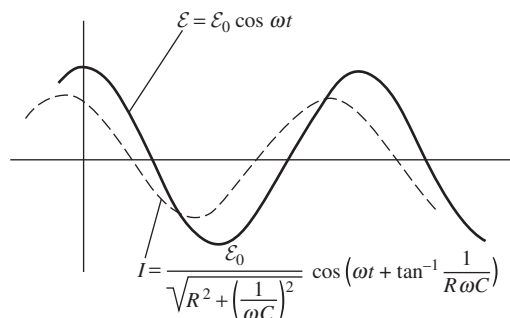


Figure 8.7.

An alternating electromotive force in a circuit containing resistance and capacitance.

Figure 8.8.

The current in the RC circuit. Compare the phase shift here with the phase shift in the inductive circuit in Fig. 8.6. The maximum in I occurs here a little earlier than the maximum in $\mathcal{E}$.

8.2.3 Transients

Mathematically speaking, the solution for the RL circuit,

$$I(t) = \frac{\mathcal{E}_0}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t - \tan^{-1} \frac{\omega L}{R}\right), \quad (8.31)$$

is a *particular integral* of the differential equation, Eq. (8.17). To this could be added a *complementary function*, that is, any solution of the homogeneous differential equation,

$$L \frac{dI}{dt} + RI = 0. \quad (8.32)$$

This is true because Eq. (8.17) is linear in I , so the superposition of the particular and complementary functions is still a solution; the complementary function simply increases the right-hand side of Eq. (8.17) by zero, and therefore doesn't affect the equality. Now, Eq. (8.32) is just Eq. (7.70) of Chapter 7, whose solution we found, in Section 7.9, to be an exponentially decaying function,

$$I(t) \sim e^{-(R/L)t}. \quad (8.33)$$

The physical significance is this: a transient, determined by some initial conditions, is represented by a decaying component of $I(t)$, of the form of Eq. (8.33). After a time $t \gg L/R$, this will have vanished, leaving only the steady sinusoidal oscillation at the driving frequency, represented by the particular integral, Eq. (8.31). This oscillation is entirely independent of the initial conditions; all memory of the initial conditions is lost.

8.2.4 RLC circuit

To solve for the current in a series RLC circuit, a certain observation will be helpful. The similarity of our results for the RL circuit and the RC circuit suggests a way to look at the inductor and capacitor in series. Suppose an alternating current $I = I_0 \cos(\omega t + \phi)$ is somehow caused to flow through such a combination (shown in Fig. 8.9). The voltage across the inductor, V_L , will be

$$V_L = L \frac{dI}{dt} = -I_0 \omega L \sin(\omega t + \phi). \quad (8.34)$$

The voltage V_C across the capacitor, with sign consistent with the sign of V_L , is

$$V_C = -\frac{Q}{C} = \frac{1}{C} \int I dt = \frac{I_0}{\omega C} \sin(\omega t + \phi). \quad (8.35)$$

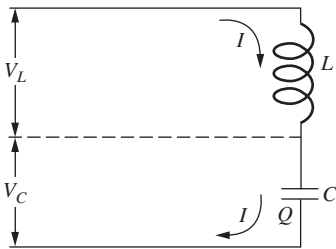


Figure 8.9.

The inductor and capacitor in series are equivalent to a single reactive element that is either an inductor or a capacitor, depending on whether $\omega^2 LC$ is greater or less than 1.

The voltage across the combination is then

$$V_L + V_C = -\left(\omega L - \frac{1}{\omega C}\right) I_0 \sin(\omega t + \phi). \quad (8.36)$$

For a given ω , the combination is evidently equivalent to a single element, either an inductor or a capacitor, depending on whether the quantity $\omega L - 1/\omega C$ is positive or negative. Suppose, for example, that $\omega L > 1/\omega C$. Then the combination is equivalent to an inductor L' such that

$$\omega L' = \omega L - \frac{1}{\omega C}. \quad (8.37)$$

Equivalence means *only* that the relation between current and voltage, for steady oscillation at the particular frequency ω , is the same. This allows us to replace L and C by L' in any circuit driven at this frequency. The main point here is that the voltages across the inductor and capacitor are both proportional to $\sin(\omega t + \phi)$, so they are always in phase with each other (or rather, exactly out of phase).

This can be applied to the simple RLC circuit in Fig. 8.10. We need only recall Eqs. (8.21) and (8.25), the solution for the RL circuit driven by the electromotive force $\mathcal{E}_0 \cos \omega t$, and replace ωL by $\omega L - 1/\omega C$:

$$I(t) = \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \cos(\omega t + \phi) \quad (8.38)$$

where

$$\tan \phi = \frac{1}{R\omega C} - \frac{\omega L}{R} \quad (8.39)$$

These expressions are also correct if $1/\omega C > \omega L$, in which case we equivalently have a capacitor C' such that $1/\omega C' = 1/\omega C - \omega L$.

Of course, we could have just solved the RLC circuit from scratch. The loop equation is

$$L \frac{dI}{dt} - \frac{Q}{C} + RI = \mathcal{E}_0 \cos \omega t. \quad (8.40)$$

Instead of either Eq. (8.19) or Eq. (8.29), we now have all three types of terms (involving L , C , and R) on the left-hand side. The coefficient of the $\sin(\omega t + \phi)$ term is $-I_0(\omega L - 1/\omega C)$, so we see that we can simply use our results for the RL circuit, with ωL replaced by $\omega L - 1/\omega C$, as we observed above.

8.2.5 Resonance

For fixed amplitude $\mathcal{E}_0$ of the electromotive force, and for given circuit elements L , C , and R , Eq. (8.38) tells us that we get the greatest current when the driving frequency ω is such that

$$\omega L - \frac{1}{\omega C} = 0, \quad (8.41)$$

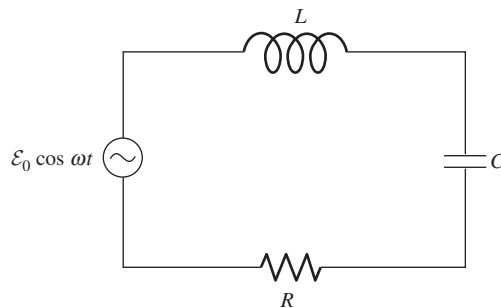


Figure 8.10. The RLC circuit driven by a sinusoidal electromotive force.

which is the same as saying that $\omega = 1/\sqrt{LC} = \omega_0$, the resonant frequency of the undamped LC circuit. In that case Eq. (8.38) reduces to

$$I(t) = \frac{\mathcal{E}_0 \cos \omega t}{R}. \quad (8.42)$$

That is exactly the current that would flow if the circuit contained the resistor alone. The reason for this is that when $\omega = 1/\sqrt{LC}$, the voltages across the inductor and capacitor are always equal and opposite. Since they cancel, they are effectively not present, and we simply have a circuit consisting of a resistor and the applied emf $\mathcal{E}_0 \cos \omega t$.

Example Consider the circuit of Fig. 8.4(a), connected now to a source or generator of alternating emf, $\mathcal{E} = \mathcal{E}_0 \cos \omega t$. The driving frequency ω may be different from the resonant frequency $\omega_0 = 1/\sqrt{LC}$, which, for the given capacitance (0.01 microfarads) and inductance (100 microhenrys), is 10^6 radians/s (or $10^6/2\pi$ cycles per second). Figure 8.11 shows the amplitude of the oscillating current as a function of the driving frequency ω , for three different values of the circuit resistance R . It is assumed that the amplitude $\mathcal{E}_0$ of the emf is 100 volts in each case. Note the resonance peak at $\omega = \omega_0$, which is most prominent and sharp for the lowest resistance value, $R = 20$ ohms. This is the same value of R for which, running as a damped oscillator without any driving emf, the circuit behaved as shown in the top graph of Fig. 8.4(b).

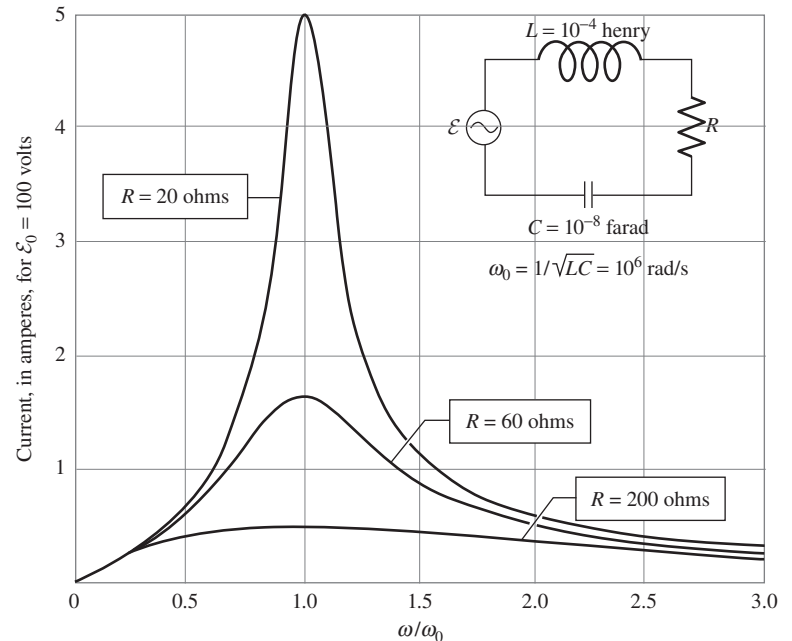


Figure 8.11.

An emf of 100 volts amplitude is applied to a series RLC circuit. The circuit elements are the same as in the example of the damped circuit in Fig. 8.4. Circuit amplitude is calculated by Eq. (8.38) and plotted, as a function of ω/ω_0 , for three different resistance values.

Note that we have encountered three (generally different) frequencies up to this point:

- the frequency of the applied oscillating emf, which can take on any value we choose;
- the resonant frequency, $\omega_0 = 1/\sqrt{LC}$, for which the amplitude of the oscillating current is largest;
- the frequency (in the underdamped case) of the transient behavior, given by Eq. (8.9). For light damping, this frequency is approximately equal to the resonant frequency, $\omega_0 = 1/\sqrt{LC}$.

8.2.6 Width of the $I_0(\omega)$ curve

The Q factor of the circuit in the above example with $R = 20$ ohms, given in Eq. (8.13) as³ $\omega_0 L/R$, is $(10^6 \cdot 10^{-4})/20$, or 5, in this case. Generally speaking, the higher the Q of a circuit, the narrower and higher the peak of its response as a function of driving frequency ω . To be more precise, consider frequencies in the neighborhood of ω_0 , writing $\omega = \omega_0 + \Delta\omega$. Then, to first order in $\Delta\omega/\omega_0$, the expression $\omega L - 1/\omega C$ that occurs in the denominator in Eq. (8.38) can be approximated this way:

$$\omega L - \frac{1}{\omega C} = \omega_0 L \left(1 + \frac{\Delta\omega}{\omega_0} \right) - \frac{1}{\omega_0 C (1 + \Delta\omega/\omega_0)}, \quad (8.43)$$

and since ω_0 is $1/\sqrt{LC}$, this becomes

$$\omega_0 L \left(1 + \frac{\Delta\omega}{\omega_0} - \frac{1}{1 + \Delta\omega/\omega_0} \right) \approx \omega_0 L \left(2 \frac{\Delta\omega}{\omega_0} \right), \quad (8.44)$$

where we have used the approximation, $1/(1 + \epsilon) \approx 1 - \epsilon$. Exactly at resonance, the quantity inside the square root sign in Eq. (8.38) is just R^2 . As ω is shifted away from resonance, the quantity under the square root will have doubled when $|\omega L - 1/\omega C| = R$, or when, approximately,

$$\frac{2|\Delta\omega|}{\omega_0} = \frac{R}{\omega_0 L} = \frac{1}{Q}. \quad (8.45)$$

This means that the current amplitude will have fallen to $1/\sqrt{2}$ times the peak when $|\Delta\omega|/\omega_0 = 1/2Q$. These are the “half-power” points, because the energy or power is proportional to the amplitude squared, as we shall explain in Section 8.6. One often expresses the width of a resonance peak by giving the full width, $2\Delta\omega$, between half-power points. Evidently that is just $1/Q$ times the resonant frequency itself. Circuits with very much higher Q than this one are quite common. A radio receiver may select a particular station and discriminate against others

³ The ω in Eq. (8.13) is the frequency of the freely decaying damped oscillator, practically the same as ω_0 for moderate or light damping. We use ω_0 here in the expression for Q . In the present discussion, ω is *any* frequency we may choose to apply to this circuit.

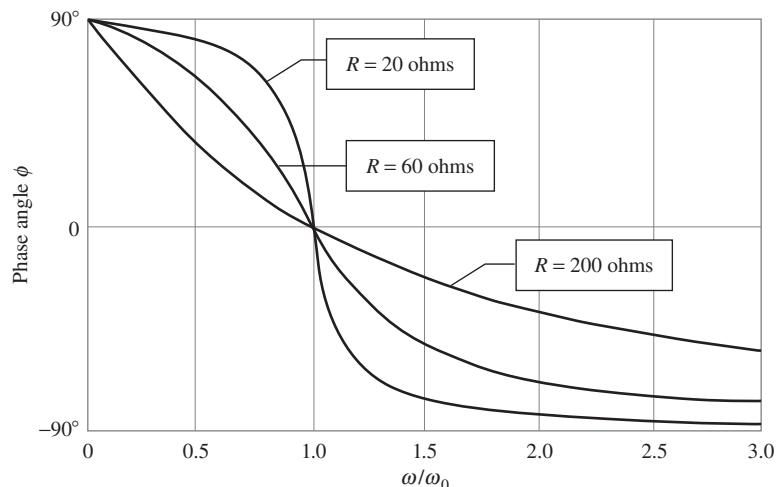


Figure 8.12.

The variation of phase angle with frequency, in the circuit of Fig. 8.11.

by means of a resonant circuit with a Q of several hundred. It is quite easy to make a microwave resonant circuit with a Q of 10^4 , or even 10^5 .

The angle ϕ , which expresses the relative phase of the current and emf oscillations, varies with frequency in the manner shown in Fig. 8.12. At a very low frequency the capacitor is the dominant hindrance to current flow, and ϕ is positive. At resonance, $\phi = 0$. The higher the Q , the more abruptly ϕ shifts from positive to negative angles as the frequency is raised through ω_0 .

To summarize what we know about Q , we have encountered two different meanings:

- In an RLC circuit with an applied oscillating emf, $1/Q$ gives a measure of the width of the current and power curves, as functions of ω . The higher the Q , the narrower the curves. More precisely, the width (at half maximum) of the power curve is ω_0/Q .
- If we remove the emf source, the current and energy will decay; Q gives a measure of how slow this decay is. The higher the Q , the more oscillations it takes for the amplitude to decrease by a given factor. More precisely, the energy decreases by a factor $1/e$ after Q radians (or $Q/2\pi$ cycles). Equivalently, as Exercise 8.17 shows, the current decreases by a factor of $e^{-\pi}$ after Q cycles. (It's hard to pass up a chance to mention a result of $e^{-\pi}$!)

8.3 Complex exponential solutions

In Section 8.2 we solved for the current in the series RLC circuit (including a voltage source $\mathcal{E}_0 \cos \omega t$) in Fig. 8.10 by guessing a sinusoidal form for the current $I(t)$. In the present section we will solve for the current in a different way, using complex numbers. This method is extremely

powerful, and it forms the basis of what we will do in the remainder of this chapter.

Our strategy will be the following. We will write down the Kirchhoff loop equation as we did above, but instead of solving it directly, we will solve a slightly modified equation in which the $\mathcal{E}_0 \cos \omega t$ voltage source is replaced by $\mathcal{E}_0 e^{i\omega t}$. We will guess an exponential solution of the form $\tilde{I}(t) = \tilde{I} e^{i\omega t}$ and solve for $\tilde{I}$, which will turn out to be a complex number.⁴ Of course, our solution for $\tilde{I}(t)$ cannot possibly be the current we are looking for, because $\tilde{I}(t)$ is complex, whereas an actual current must be real. However, if we take the real part of $\tilde{I}(t)$, we will obtain (for reasons we will explain) the desired current $I(t)$ that actually flows in the circuit. Let's see how all this works. Our goal is to reproduce the $I(t)$ in Eqs. (8.38) and (8.39).

The Kirchhoff loop equation for the series RLC circuit in Fig. 8.10 is⁵

$$L \frac{dI(t)}{dt} + RI(t) + \frac{Q(t)}{C} = \mathcal{E}_0 \cos \omega t. \quad (8.46)$$

If we take clockwise current to be positive, then $Q(t)$ is the integral of $I(t)$, that is, $Q(t) = \int I(t) dt$. Consider now the modified equation where $\cos \omega t$ is replaced by $e^{i\omega t}$,

$$L \frac{d\tilde{I}(t)}{dt} + R\tilde{I}(t) + \frac{\tilde{Q}(t)}{C} = \mathcal{E}_0 e^{i\omega t}. \quad (8.47)$$

If $\tilde{I}(t)$ is a (complex) solution to this equation, then if we take the real part of the entire equation, we obtain (using the facts that differentiation and integration with respect to t commute with taking the real part)

$$L \frac{d}{dt} \text{Re}[\tilde{I}(t)] + R \text{Re}[\tilde{I}(t)] + \frac{1}{C} \int \text{Re}[\tilde{I}(t)] dt = \mathcal{E}_0 \cos \omega t. \quad (8.48)$$

We have used the remarkable mathematical identity, $e^{i\theta} = \cos \theta + i \sin \theta$, which tells us that $\cos \omega t$ is the real part of $e^{i\omega t}$. (See Appendix K for a review of complex numbers.)

Equation (8.48) is simply the statement that $I(t) \equiv \text{Re}[\tilde{I}(t)]$ is a solution to our original differential equation in Eq. (8.46). Our goal is therefore to find a complex function $\tilde{I}(t)$ that satisfies Eq. (8.47), and then take the real part. Note the critical role that linearity played here.

⁴ The tilde on the I terms denotes a complex number. Note that $\tilde{I}(t)$ has time dependence, whereas $\tilde{I}$ does not. More precisely, $\tilde{I} = \tilde{I}(0)$. When writing $\tilde{I}(t)$, be careful not to drop the t argument, because that will change the meaning to $\tilde{I}$ (although the meaning is generally clear from the context). There will actually be a total of four different versions of the letter I that we will encounter in this method. They are summarized in Fig. 8.13.

⁵ We are now taking Q to be the charge on the top plate of the capacitor (for no deep reason). You should verify that if we instead took Q to be the charge on the bottom plate, then two minus signs would end up canceling, and we would still arrive at Eq. (8.48). After all, that equation for $\tilde{I}(t)$ can't depend on our arbitrary convention for Q .

If our differential equation were modified to contain a term that wasn't linear in $I(t)$, for example $RI(t)^2$, then this method wouldn't work, because $\text{Re}[\tilde{I}(t)^2]$ is *not* equal to $(\text{Re}[\tilde{I}(t)])^2$. The modified form of Eq. (8.48) would *not* be the statement that $I(t) \equiv \text{Re}[\tilde{I}(t)]$ satisfies the modified form of Eq. (8.46).

A function of the form $\tilde{I}(t) = \tilde{I}e^{i\omega t}$ will certainly yield a solution to Eq. (8.47), because the $e^{i\omega t}$ factor will cancel through the whole equation, yielding an equation with no time dependence. Now, if $\tilde{I}(t) = \tilde{I}e^{i\omega t}$, then $\tilde{Q}(t)$, which is the integral of $\tilde{I}(t)$, equals $\tilde{I}e^{i\omega t}/i\omega$. (There is no need for a constant of integration because we know that Q oscillates around zero.) So Eq. (8.47) becomes

$$Li\omega\tilde{I}e^{i\omega t} + R\tilde{I}e^{i\omega t} + \frac{\tilde{I}e^{i\omega t}}{i\omega C} = \mathcal{E}_0e^{i\omega t}. \quad (8.49)$$

Canceling the $e^{i\omega t}$, solving for $\tilde{I}$, and getting the i out of the denominator by multiplying by 1 in the form of the complex conjugate divided by itself, yields

$$\tilde{I} = \frac{\mathcal{E}_0}{i\omega L + R + 1/i\omega C} = \frac{\mathcal{E}_0[R - i(\omega L - 1/\omega C)]}{R^2 + (\omega L - 1/\omega C)^2}. \quad (8.50)$$

The term in the square brackets is a complex number written in $a + bi$ form, but it will be advantageous to write it in “polar” form, that is, as a magnitude times a phase, $Ae^{i\phi}$. The magnitude is $A = \sqrt{a^2 + b^2}$, and the phase is $\phi = \tan^{-1}(b/a)$; see Problem 8.7. So we have

$$\begin{aligned} \tilde{I} &= \frac{\mathcal{E}_0}{R^2 + (\omega L - 1/\omega C)^2} \cdot \sqrt{R^2 + (\omega L - 1/\omega C)^2} e^{i\phi} \\ &= \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} e^{i\phi} \equiv I_0 e^{i\phi}, \end{aligned} \quad (8.51)$$

where

$$I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \quad \text{and} \quad \tan \phi = \frac{1}{R\omega C} - \frac{\omega L}{R}. \quad (8.52)$$

The actual current $I(t)$ is obtained by taking the real part of the full $\tilde{I}(t) = \tilde{I}e^{i\omega t}$ solution:

$$\begin{aligned} I(t) &= \text{Re}[\tilde{I}e^{i\omega t}] = \text{Re}[I_0 e^{i\phi} e^{i\omega t}] = I_0 \cos(\omega t + \phi) \\ &= \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \cos(\omega t + \phi), \end{aligned} \quad (8.53)$$

in agreement with Eqs. (8.38) and (8.39). I_0 is the amplitude of the current, and ϕ is the phase relative to the applied voltage.

As mentioned above, there are four different types of I 's that appear in this procedure: $\tilde{I}(t)$, $\tilde{I}$, $I(t)$, and I_0 . These are related to each other in the following ways (summarized in Fig. 8.13).

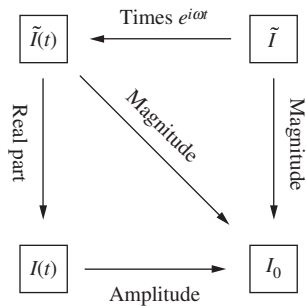


Figure 8.13. Relations among the various usages of the letter “I.”

- The two complex quantities, $\tilde{I}(t)$ and $\tilde{I}$, are related by a simple factor of $e^{i\omega t}$: $\tilde{I}(t) = \tilde{I}e^{i\omega t}$; $\tilde{I}$ equals $\tilde{I}(0)$.
- $I(t)$, which is the actual current, equals the real part of $\tilde{I}(t)$: $I(t) = \text{Re}[\tilde{I}(t)]$.
- I_0 is the magnitude of both $\tilde{I}(t)$ and $\tilde{I}$: $I_0 = |\tilde{I}(t)|$ and $I_0 = |\tilde{I}|$.
- I_0 is the amplitude of $I(t)$: $I(t) = I_0 \cos(\omega t + \phi)$.

Although the above method involving complex exponentials might take some getting used to, it is much cleaner and quicker than the method involving trig functions that we used in [Section 8.2](#). Recall the system of equations that we needed to solve in [Eqs. \(8.21\)–\(8.25\)](#). We had to demand that the coefficients of $\sin \omega t$ and $\cos \omega t$ in [Eq. \(8.20\)](#) were independently zero. That involved a fair bit of algebra. In the present complex-exponential method, the $e^{i\omega t}$ terms cancel in [Eq. \(8.49\)](#), so we are left with only one equation, which we can quickly solve. The point here is that the derivative of an exponential gives back an exponential, whereas sines and cosines flip flop under differentiation. Of course, from the relation $e^{i\theta} = \cos \theta + i \sin \theta$, we know that exponentials can be written in terms of trig functions, and vice versa via $\cos \theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin \theta = (e^{i\theta} - e^{-i\theta})/2i$. So any task that can be accomplished with exponential functions can also be accomplished with trig functions. But exponentials invariably make the calculations much easier.

In the event that the applied voltage isn't a nice sinusoidal function, our method of guessing exponentials (or trig functions) is still applicable, due to two critical things: (1) Fourier analysis and (2) the linearity of the differential equation in [Eq. \(8.46\)](#). You will study the all-important subject of Fourier analysis in your future math and physics courses, but for now we simply note that Fourier analysis tells us that any reasonably well-behaved function for the voltage source can be written as the (perhaps infinite) sum of exponentials, or equivalently trig functions. And then linearity tells us that we can just add up the solutions for all these exponential voltage sources to obtain the solution for the original voltage source. In effect, this is what we did when we took the real part of $\tilde{I}(t)$ to obtain the actual current $I(t)$. We would have arrived at the same answer if we wrote the applied voltage $\mathcal{E}_0 \cos \omega t$ as $\mathcal{E}_0(e^{i\omega t} + e^{-i\omega t})/2$, then found the solutions for these two exponential voltages, and then added them together. So the strategy of taking the real part is just a special case of the strategy of superposing solutions via Fourier analysis.

8.4 Alternating-current networks

In this section we will generalize the results from [Section 8.3](#), where our circuit involved only one loop. Complex numbers provide us with a remarkably efficient way of dealing with arbitrary alternating-current networks. An alternating-current network is any collection of resistors,

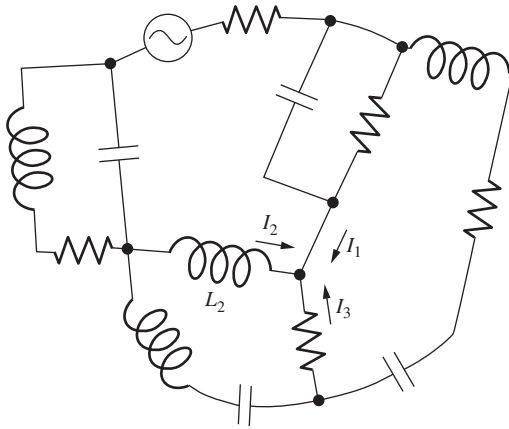


Figure 8.14.
An alternating-current network.

capacitors, and inductors in which currents flow that are oscillating steadily at the constant frequency ω . One or more electromotive forces, at this frequency, drive the oscillation. Figure 8.14 is a diagram of one such network. The source of alternating electromotive force is represented by the symbol $\sim$. In a branch of the network, for instance the branch that contains the inductor L_2 , the current as a function of time is

$$I_2(t) = I_{02} \cos(\omega t + \phi_2). \quad (8.54)$$

Since the frequency is a constant for the whole network, two numbers, such as the amplitude I_{02} and the phase constant ϕ_2 above, are enough to determine for all time the current in a particular branch. Similarly, the voltage across a branch oscillates with a certain amplitude and phase:

$$V_2(t) = V_{02} \cos(\omega t + \theta_2). \quad (8.55)$$

If we have determined the currents and voltages in all branches of a network, we have analyzed it completely. To find them by constructing and solving all the appropriate differential equations is possible, of course; and if we were concerned with the transient behavior of the network, we might have to do something like that. For the steady state at some given frequency ω , we can use a far simpler and more elegant method. It is based on two ideas:

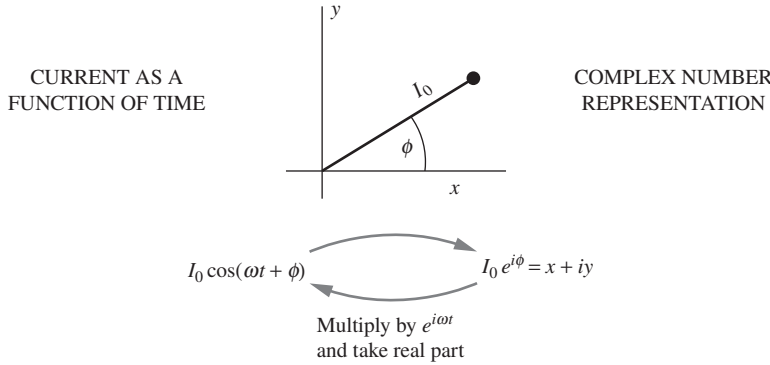
- (1) An alternating current or voltage can be represented by a complex number;
- (2) Any one branch or element of the circuit can be characterized, at a given frequency, by the relation between the voltage and current in that branch.

As we saw above, the first idea exploits the identity, $e^{i\theta} = \cos \theta + i \sin \theta$. To carry it out we adopt the following *rule* for the representation:

An alternating current $I(t) = I_0 \cos(\omega t + \phi)$ is to be *represented* by the complex number $I_0 e^{i\phi}$, that is, the number whose real part is $I_0 \cos \phi$ and whose imaginary part is $I_0 \sin \phi$.

Going the other way, if the complex number $x + iy$ *represents* a current $I(t)$, then the current as a function of time is given by the real part of the product $(x + iy)e^{i\omega t}$. Equivalently, if $I_0 e^{i\phi}$ represents a current $I(t)$, then $I(t)$ is given by the real part of the product $I_0 e^{i\phi} e^{i\omega t}$, which is $I_0 \cos(\omega t + \phi)$.

Figure 8.15 is a reminder of this two-way correspondence. Since a complex number $z = x + iy$ can be graphically represented on the two-dimensional plane, it is easy to visualize the phase constant as the angle $\tan^{-1}(y/x)$ and the amplitude I_0 as the modulus $\sqrt{x^2 + y^2}$.

**Figure 8.15.**

Rules for representing an alternating current by a complex number.

What makes all this useful is the following fact. *The representation of the sum of two currents is the sum of their representations.* Consider the sum of two currents $I_1(t)$ and $I_2(t)$ that meet at a junction of wires in Fig. 8.14. At any instant of time t , the sum of the currents is given by

$$\begin{aligned} I_1(t) + I_2(t) &= I_{01} \cos(\omega t + \phi_1) + I_{02} \cos(\omega t + \phi_2) \\ &= (I_{01} \cos \phi_1 + I_{02} \cos \phi_2) \cos \omega t \\ &\quad - (I_{01} \sin \phi_1 + I_{02} \sin \phi_2) \sin \omega t. \end{aligned} \quad (8.56)$$

On the other hand, the sum of the complex numbers that, according to our rule, represent $I_1(t)$ and $I_2(t)$ is

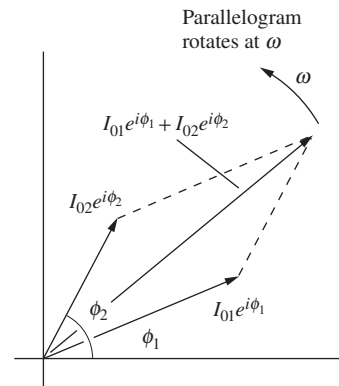
$$I_{01} e^{i\phi_1} + I_{02} e^{i\phi_2} = (I_{01} \cos \phi_1 + I_{02} \cos \phi_2) + i(I_{01} \sin \phi_1 + I_{02} \sin \phi_2). \quad (8.57)$$

If you multiply the right-hand side of Eq. (8.57) by $\cos \omega t + i \sin \omega t$ and take the real part of the result, you will get just what appears on the right in Eq. (8.56). This is no surprise, of course, because what we've just done is show (the long way) that

$$\operatorname{Re} \left[I_{01} e^{i(\omega t + \phi_1)} + I_{02} e^{i(\omega t + \phi_2)} \right] = \operatorname{Re} \left[(I_{01} e^{i\phi_1} + I_{02} e^{i\phi_2}) (e^{i\omega t}) \right]. \quad (8.58)$$

The left-hand side of this equation is what appears in Eq. (8.56), and the right-hand side is the result of multiplying Eq. (8.57) by $e^{i\omega t} = \cos \omega t + i \sin \omega t$ and taking the real part.

Figure 8.16 shows geometrically what is going on. The real part of a number in the complex plane is its projection onto the x axis. So the current $I_1(t) = I_{01} \cos(\omega t + \phi_1)$ is the horizontal projection of the complex number $I_{01} e^{i(\omega t + \phi_1)}$, and this complex number can be visualized as the vector $I_{01} e^{i\phi_1}$ rotating around in the plane with angular frequency ω (because the angle increases according to ωt). Likewise for the current $I_2(t) = I_{02} \cos(\omega t + \phi_2)$. Now, *the sum of the projections of two vectors is*

**Figure 8.16.**

As these three vectors rotate around in the plane with the same frequency ω , the horizontal projection of the long vector (the sum) always equals the sum of the horizontal projections of the other two vectors.

the projection of the sum. So we can alternatively obtain the total current $I_1(t) + I_2(t)$ by finding the projection of the sum of the representations, which is the complex number $I_{01}e^{i\phi_1} + I_{02}e^{i\phi_2}$, as this sum rotates around in the plane with frequency ω . We see that the validity of the statement, “The representation of the sum of two currents is the sum of their representations,” boils down to the geometrical fact that the parallelogram in Fig. 8.16 keeps the same shape as it rotates around in the plane.

This means that, instead of adding or subtracting the periodic functions of time themselves, we can add or subtract the complex numbers that represent them. Or, putting it another way, the algebra of alternating currents turns out to be the same as the algebra of complex numbers with respect to addition. The correspondence does *not* extend to multiplication. The complex number $I_{01}I_{02}e^{i(\phi_1+\phi_2)}$ does *not* represent the product of the two current functions in Eq. (8.56), because the real part of the product of two complex numbers is not equal to the product of the real parts (the latter omits the contribution from the product of the imaginary parts).

However, it is only addition of currents and voltages that we need to carry out in analyzing the network. For example, at the junction where $I_1(t)$ meets $I_2(t)$ in Fig. 8.14, there is the physical requirement that *at every instant* the net flow of current into the junction shall be zero. Hence the condition

$$I_1(t) + I_2(t) + I_3(t) = 0 \quad (8.59)$$

must hold, where $I_1(t)$, $I_2(t)$, and $I_3(t)$ are the *actual periodic functions of time*. Thanks to our correspondence, this can be expressed in the simple algebraic statement that the sum of three complex numbers is zero. Voltages can be handled in the same way. Instantaneously, the sum of voltage drops around any loop in the network must equal the electromotive force in the loop at that instant. This condition relating periodic voltage functions can likewise be replaced by a statement about the sum of some complex numbers, the representations of the various oscillating functions, $V_1(t)$, $V_2(t)$, etc.

8.5 Admittance and impedance

The relation between current flow in a circuit element and the voltage across the element can be expressed as a relation between the complex numbers that represent the voltage and the current. Look at the inductor–resistor combination in Fig. 8.5. The voltage oscillation is represented by⁶ $\tilde{V} = \mathcal{E}_0$ and the current by $\tilde{I} = I_0e^{i\phi}$, where $I_0 = \mathcal{E}_0/\sqrt{R^2 + \omega^2L^2}$ and $\tan \phi = -\omega L/R$. The phase difference ϕ and the ratio of current

⁶ As in Section 8.3, we will indicate complex voltages (and currents) by putting a tilde over them, to avoid confusion with the actual voltages (or currents) $V(t)$ which, as we have noted, are given by the real part of $\tilde{V}e^{i\omega t}$.

amplitude to voltage amplitude are properties of the circuit at this frequency. We define a complex number Y as follows:

$$Y = \frac{e^{i\phi}}{\sqrt{R^2 + \omega^2 L^2}}, \quad \text{with} \quad \phi = \tan^{-1} \left(-\frac{\omega L}{R} \right). \quad (8.60)$$

Then the relation

$$\tilde{I} = Y \tilde{V} \quad (8.61)$$

holds, where $\tilde{V}$ is the complex number (which happens to be just the real number $\mathcal{E}_0$ in the present case) that represents the voltage across the series combination of R and L , and $\tilde{I}$ is the complex number that represents the current. Y is called the *admittance*. The same relation can be expressed with the reciprocal of Y , denoted by⁷ Z and called the *impedance*:

$$\tilde{V} = \left(\frac{1}{Y} \right) \tilde{I} \Rightarrow \tilde{V} = Z \tilde{I} \quad (8.62)$$

In Eqs. (8.61) and (8.62) we do make use of the product of two complex numbers, but only one of the numbers is the representation of an alternating current or voltage. The other is the impedance or admittance. Our algebra thus contains two categories of complex numbers, those that represent admittances and impedances, and those that represent currents and voltages. The product of two “impedance numbers,” like the product of two “current numbers,” doesn’t represent anything.

The impedance is measured in ohms. Indeed, if the circuit element had consisted of the resistance R alone, the impedance would be real and equal simply to R , so that Eq. (8.62) would resemble Ohm’s law for a direct-current circuit: $V = RI$.

The admittance of a resistanceless inductor is the imaginary quantity $Y = -i/\omega L$. This can be seen by letting R go to zero in Eq. (8.60), which yields $\phi = -\pi/2 \Rightarrow e^{i\phi} = -i$. The factor $-i$ means that the current oscillation lags the voltage oscillation by $\pi/2$ in phase. On the complex number diagram, if the voltage is represented by $\tilde{V}$ (Fig. 8.17(b)), the current might be represented by $\tilde{I}$, located as shown there. For the capacitor, we have $Y = i\omega C$, as can be seen from the expression for the current in Eq. (8.30). In this case $\tilde{V}$ and $\tilde{I}$ are related as indicated in Fig. 8.17(c); the current leads the voltage by $\pi/2$. The inset in each of the figures shows how the relative sign of $\tilde{V}$ and $\tilde{I}$ is to be specified. Unless that is done consistently, *leading* and *lagging* are meaningless. Note that we always define the positive current direction so that a positive voltage applied to a

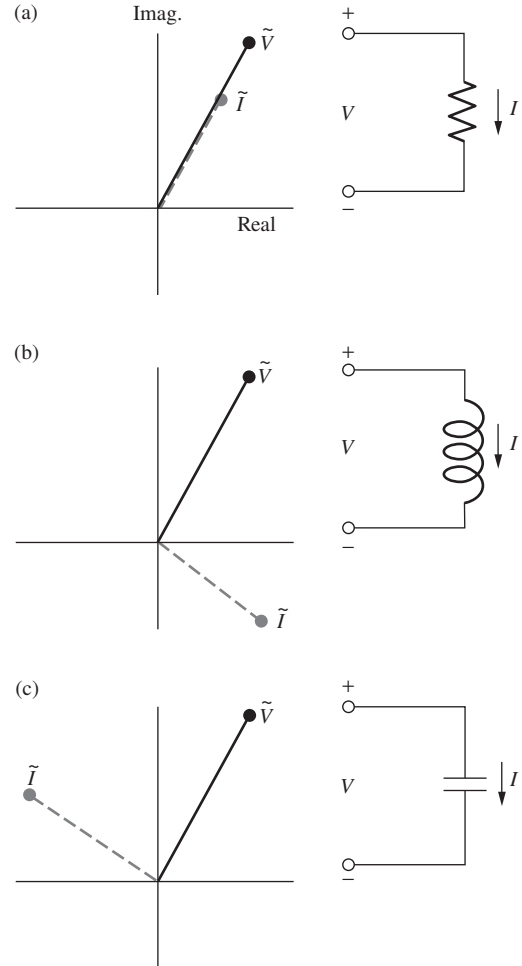


Figure 8.17.

$\tilde{V}$ and $\tilde{I}$ are complex numbers that represent the voltage across a circuit element and the current through it. The relative phase of current and voltage oscillation is manifest here in the angle between the “vectors.” (a) In the resistor, current and voltage are in phase. (b) In the inductor, current lags the voltage. (c) In the capacitor, current leads the voltage.

⁷ We won’t put a tilde over Y or Z , even though they are complex numbers, because we will rarely have the need to take their real parts (except when finding the phase ϕ). So we won’t need to worry about confusion between two different types of impedances.

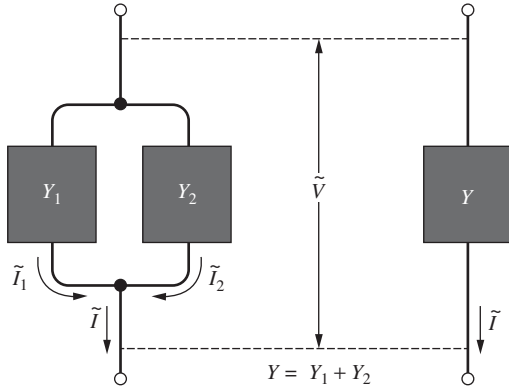


Figure 8.18.
Combining admittances in parallel.

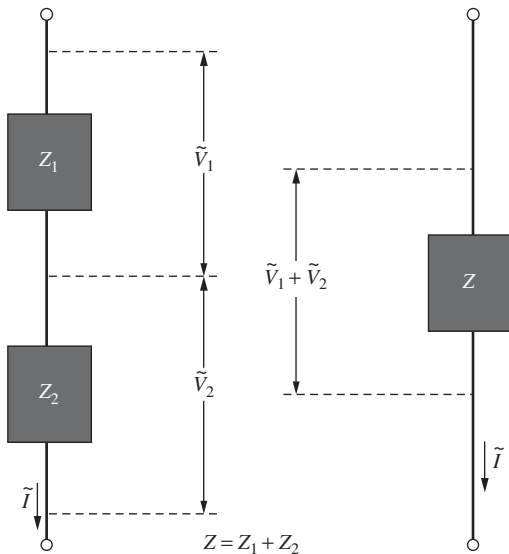


Figure 8.19.
Combining impedances in series.

Table 8.1.
Complex impedances

Symbol	Admittance, Y	Impedance, $Z = 1/Y$
R 	$\frac{1}{R}$	R
L 	$\frac{1}{i\omega L}$	$i\omega L$
C 	$i\omega C$	$\frac{1}{i\omega C}$
$I = YV$		$V = ZI$

resistor causes positive current (Fig. 8.17(a)). The properties of the three basic circuit elements are summarized in Table 8.1.

We can build up any circuit from these elements. When elements or combinations of elements are connected in *parallel*, it is convenient to use the *admittance*, for in that case admittances add. In Fig. 8.18 two black boxes with admittances Y_1 and Y_2 are connected in parallel. Since the voltages across each box are the same and since the currents add, we have

$$\tilde{I} = \tilde{I}_1 + \tilde{I}_2 = Y_1 \tilde{V} + Y_2 \tilde{V} = (Y_1 + Y_2) \tilde{V}, \quad (8.63)$$

which implies that the equivalent single black box has an admittance $Y = Y_1 + Y_2$. From Fig. 8.19 we see that the *impedances* add for elements connected in *series*, because the currents are the same and the voltages add:

$$\tilde{V} = \tilde{V}_1 + \tilde{V}_2 = Z_1 \tilde{I} + Z_2 \tilde{I} = (Z_1 + Z_2) \tilde{I}, \quad (8.64)$$

which implies that the equivalent single black box has an impedance $Z = Z_1 + Z_2$. It sounds as if we are talking about a direct-current network! In fact, we have now reduced the ac network problem to the dc network problem, with only this difference: the numbers we deal with are complex numbers.

Example (Parallel RLC circuit) Consider the “parallel RLC ” circuit in Fig. 8.20. The combined admittance of the three parallel branches is

$$Y = \frac{1}{R} + i\omega C - \frac{i}{\omega L}. \quad (8.65)$$

The voltage is simply $\mathcal{E}_0$, so the complex current is

$$\tilde{I} = Y \tilde{V} = \left[\frac{1}{R} + i \left(\omega C - \frac{1}{\omega L} \right) \right] \mathcal{E}_0. \quad (8.66)$$

The amplitude I_0 of the current oscillation $I(t)$ is the modulus of the complex number $\tilde{I}$, and the phase angle relative to the voltage is $\tan^{-1}[\text{Im}(Y)/\text{Re}(Y)]$.

Assuming that the voltage is given as usual by $\mathcal{E}_0 \cos \omega t$ (that is, with no phase), we have

$$I(t) = \mathcal{E}_0 \sqrt{(1/R)^2 + (\omega C - 1/\omega L)^2} \cos(\omega t + \phi),$$

$$\tan \phi = R\omega C - \frac{R}{\omega L}. \quad (8.67)$$

You can compare these results with the results in Eqs. (8.38) and (8.39) for the series RLC circuit. For both of these circuits, you are encouraged to check limiting cases for the R , L , and C values.

Let's now analyze a more complicated circuit. We will examine in detail what the various complex voltages and currents look like in the complex plane and how they relate to each other.

Example Consider the circuit in Fig. 8.21. Our goal will be to find the complex voltage across, and current through, each of the three elements. We will then draw the associated vectors in the complex plane and verify that the relations among them are correct. To keep the calculations from getting out of hand, we will arrange for all three of the complex impedances to have magnitude R . If we take R and ω as given, this can be arranged by letting $L = R/\omega$ and $C = 1/\omega R$. The three impedances are then

$$Z_R = R, \quad Z_L = i\omega L = iR, \quad Z_C = 1/i\omega C = -iR. \quad (8.68)$$

With these values, the impedance of the entire circuit is

$$Z = Z_C + \frac{Z_R Z_L}{Z_R + Z_L} = R \left(-i + \frac{1 \cdot i}{1 + i} \right) = R \frac{1 - i}{2}. \quad (8.69)$$

Assuming that the applied voltage is given as usual by $\mathcal{E}_0 \cos \omega t$ (with no extra phase), the applied complex voltage $\tilde{V}_{\mathcal{E}}$ is simply the real number $\mathcal{E}_0$. The total complex current $\tilde{I}$ (which is also the complex current $\tilde{I}_C$ through the capacitor) is therefore given by

$$\tilde{V}_{\mathcal{E}} = \tilde{I}Z \implies \tilde{I} = \frac{\mathcal{E}_0}{Z} = \frac{\mathcal{E}_0}{R} \frac{2}{1 - i} = \frac{\mathcal{E}_0}{R} (1 + i). \quad (8.70)$$

The complex voltage across the capacitor is then

$$\tilde{V}_C = \tilde{I}Z_C = \frac{\mathcal{E}_0}{R} (1 + i) \cdot (-iR) = \mathcal{E}_0 (1 - i). \quad (8.71)$$

The complex voltages across the resistor and inductor are the same, and their common value equals $\mathcal{E}_0$ minus the complex voltage across the capacitor:

$$\tilde{V}_R = \tilde{V}_L = \mathcal{E}_0 - \tilde{V}_C = \mathcal{E}_0 - \mathcal{E}_0 (1 - i) = i\mathcal{E}_0. \quad (8.72)$$

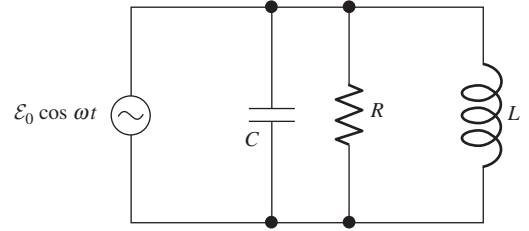


Figure 8.20.

A parallel resonant circuit. Add the complex admittances of the three elements, as in Eq. (8.65).

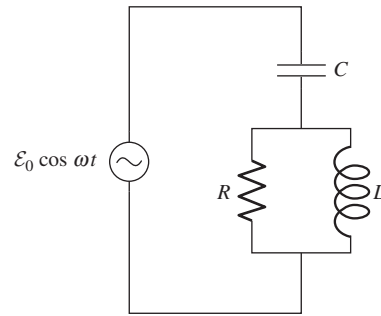


Figure 8.21.

What are the complex voltages and currents across each of the three elements in this circuit?

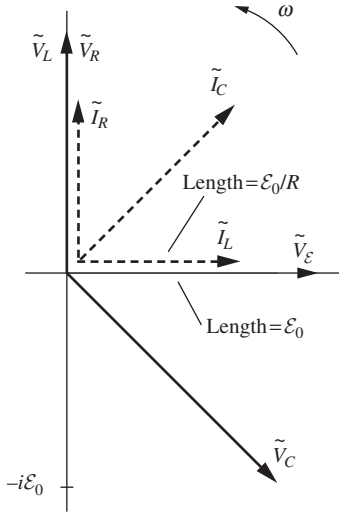


Figure 8.22.
The various complex voltages and currents for the circuit in Fig. 8.21.

The complex current through the resistor is therefore

$$\tilde{I}_R = \frac{\tilde{V}_R}{Z_R} = \frac{i\mathcal{E}_0}{R}, \quad (8.73)$$

and the complex current through the inductor is

$$\tilde{I}_L = \frac{\tilde{V}_R}{Z_L} = \frac{i\mathcal{E}_0}{iR} = \frac{\mathcal{E}_0}{R}. \quad (8.74)$$

Our results for the three complex voltages (along with the $\mathcal{E}_0$ source) and the three complex currents are drawn in the complex plane in Fig. 8.22. (The $\tilde{V}$'s and $\tilde{I}$'s have different units, so the relative size of the two groups of vectors is meaningless.) There are various true statements we can make about the vectors: (1) $\mathcal{E}_0$ equals the sum of $\tilde{V}_C$ and either $\tilde{V}_L$ or $\tilde{V}_R$, (2) $\tilde{I}_C$ equals the sum of $\tilde{I}_R$ and $\tilde{I}_L$, (3) $\tilde{I}_L$ is 90° behind $\tilde{V}_L$ as the vectors rotate counterclockwise around in the plane, (4) $\tilde{I}_R$ is in phase with $\tilde{V}_R$, and (5) $\tilde{I}_C$ is 90° ahead of $\tilde{V}_C$.

As time goes on, the vectors in Fig. 8.22 all rotate around in the complex plane with the same angular speed ω . The vectors keep the same rigid shape with respect to each other. The horizontal projections (the real parts) are the actual quantities that exist in the real world. Equivalently, the actual quantities are given by $I_R(t) = \text{Re}[\tilde{I}_R e^{i\omega t}]$, etc. The $e^{i\omega t}$ factor increases the phase by ωt , so this is what causes the vectors to rotate around in the plane. Figure 8.22 gives the vectors at $t = 0$ (assuming the applied voltage equals $\mathcal{E}_0 \cos \omega t$ with no extra phase), or at any time for which ωt is a multiple of 2π .

As mentioned in Section 8.4, the critical thing to realize about this rotation around in the plane is that since, for example, the vector $\tilde{I}_C$ always equals the sum of vectors $\tilde{I}_R$ and $\tilde{I}_L$ (because the system rotates as a rigid “object”), the horizontal projections also always satisfy this relation. That is, $I_C(t) = I_R(t) + I_L(t)$. In other words, the Kirchhoff node condition is satisfied at the node below the capacitor. Likewise, since the applied voltage $\tilde{V}_E$ always equals $\tilde{V}_C$ plus $\tilde{V}_R$ (or $\tilde{V}_L$), we have $V_E(t) = V_C(t) + V_R(t)$. So the Kirchhoff loop condition is satisfied. In short, if the complex voltages and currents satisfy Kirchhoff's rules at a *particular* time, then the actual voltages and currents satisfy Kirchhoff's rules at *all* times.

As noted earlier in this section, the i 's in Z_L and Z_C in Table 8.1 are consistent with the $\pm\pi/2$ phases between the voltages and currents. Let's verify this for Fig. 8.22. In the case of the inductor, we have

$$\tilde{V}_L = \tilde{I}_L Z_L \implies \tilde{V}_L = \tilde{I}_L (i\omega L) \implies \tilde{V}_L = \tilde{I}_L (e^{i\pi/2} \omega L), \quad (8.75)$$

which means that $\tilde{V}_L$ is $\pi/2$ ahead of $\tilde{I}_L$. The opposite is true for the capacitor. More generally, we can write $\tilde{V} = \tilde{I}Z$ for the entire circuit or any subpart, just as we can for a network containing only resistors.

If the complex voltage $\tilde{V}$, complex current $\tilde{I}$, and impedance Z are written in polar form as⁸

$$\tilde{V} = V_0 e^{i\phi_V}, \quad \tilde{I} = I_0 e^{i\phi_I}, \quad Z = |Z| e^{i\phi_Z}, \quad (8.76)$$

then, by looking at the modulus and phase of the two sides of the $\tilde{V} = \tilde{I}Z$ equation, we obtain

$$\boxed{V_0 = I_0 |Z|} \quad \text{and} \quad \boxed{\phi_V = \phi_I + \phi_Z} \quad (8.77)$$

The former of these statements looks just like Ohm's law, $V = IR$. The latter says that the voltage is ϕ_Z ahead of the current. You are encouraged at this point to solve Problem 8.9, the task of which is to draw all the complex voltages and currents for the series and parallel RLC circuits in Figs. 8.10 and 8.20.

We should emphasize that the above methods are valid only for *linear* circuit elements, elements in which the current is proportional to the voltage. In other words, our circuit must be described by a linear differential equation. You can't even define an impedance for a nonlinear element. Nonlinear circuit elements are very important and interesting devices. If you have studied some in the laboratory, you can see why they will not yield to this kind of analysis.

This is all predicated, too, on continuous oscillation at constant frequency. The transient behavior of the circuit is a different problem. However, for linear circuits the tools we have just developed have some utility, even for transients. The reason, as we noted at the end of Section 8.3, is that by superposing steady oscillations of many frequencies we can represent a nonsteady behavior, and the response to each of the individual frequencies can be calculated as if that frequency were present alone.

We have encountered three different methods for dealing with steady states in circuits containing a sinusoidal voltage source. Let's summarize them.

Method 1 (Trig functions)

This is the method we used in Section 8.2. The steps are as follows.

- Write down the differential equation expressing the fact that the voltage drop around each loop in a circuit is zero. The various voltage drops take the form of IR , $L di/dt$, and Q/C . Write the differential equation in terms of only one quantity, say the current $I(t)$.
- Guess a trig solution of the form $I(t) = I_0 \cos(\omega t + \phi)$. There will be many such currents if there are many loops.

⁸ We have written the modulus of Z as $|Z|$ rather than Z_0 to signify that Z isn't the same type of quantity as $\tilde{V}$ and $\tilde{I}$. The quantities V_0 and I_0 are the amplitudes of the actual voltage and current oscillations, and we don't want to give the impression that Z represents an oscillatory function.

- Use the trig sum formulas to expand $\cos(\omega t + \phi)$ and $\sin(\omega t + \phi)$, and then demand that the coefficients of $\cos \omega t$ and $\sin \omega t$ are separately identically equal to zero. This yields solutions for I_0 and ϕ .

Method 2 (Exponential functions)

This is the method we used in [Section 8.3](#). The steps are as follows.

- As in Method 1, write down the differential equation for the voltage drop around each loop, and then write it in terms of only, say, the current $I(t)$.
- Replace the $\mathcal{E}_0 \cos \omega t$ voltage source with $\mathcal{E}_0 e^{i\omega t}$, and then guess a complex solution for the current of the form $\tilde{I}(t) \equiv \tilde{I} e^{i\omega t}$. The actual current in the circuit will be given by the real part of this. That is, $I(t) = \text{Re}[\tilde{I}(t)]$. There will be many such currents if there are many loops.
- The solution for $\tilde{I}$ can be written in the general polar form, $\tilde{I} = I_0 e^{i\phi}$. The actual current is then

$$I(t) = \text{Re}[\tilde{I}(t)] = \text{Re}[\tilde{I} e^{i\omega t}] = \text{Re}[I_0 e^{i\phi} e^{i\omega t}] = I_0 \cos(\omega t + \phi). \quad (8.78)$$

I_0 is the amplitude of the current, and ϕ is the phase relative to the voltage source.

Method 3 (Complex impedances)

This is the method we used in [Sections 8.4](#) and [8.5](#). The steps are as follows.

- Assign impedances of R , $i\omega L$, and $1/i\omega C$ to the resistors, inductors, and capacitors in the circuit, and then use the standard rules for adding impedances in series and in parallel (the same rules as for simple resistors).
- Write down $\tilde{V} = \tilde{I}Z$ for the entire circuit or any subpart, just as you would for a network containing only resistors. With the complex quantities written in polar form, $\tilde{V} = \tilde{I}Z$ quickly yields $V_0 = I_0|Z|$ and $\phi_V = \phi_I + \phi_Z$. The former of these statements looks just like Ohm's law, $V = IR$. The latter says that the voltage is ϕ_Z ahead of the current.
- The $\tilde{V}$ and $\tilde{I}$ vectors rotate around in the complex plane with the same angular speed ω . The horizontal projections (the real parts) are the actual quantities that exist in the real world. Since the vectors keep the same rigid shape with respect to each other, it follows that if the complex voltages and currents satisfy Kirchhoff's rules at a given time, the actual voltages and currents satisfy Kirchhoff's rules at all times.
- This third method is actually just a more systematic version of the second method. But for circuits involving more than one loop, the third method is vastly more tractable than the second, which in turn is much more tractable than the first.

8.6 Power and energy in alternating-current circuits

If the voltage across a resistor R is $V_0 \cos \omega t$, the current is $I = (V_0/R) \cos \omega t$. The instantaneous power, that is, the instantaneous rate at which energy is being dissipated in the resistor, is given by

$$P_R = RI^2 = \frac{V_0^2}{R} \cos^2 \omega t. \quad (8.79)$$

Since the average of $\cos^2 \omega t$ over many cycles is $1/2$ (because it has the same average as $\sin^2 \omega t$, and $\sin^2 \omega t + \cos^2 \omega t = 1$), the average power dissipated in the resistor is

$$\overline{P}_R = \frac{1}{2} \frac{V_0^2}{R}. \quad (8.80)$$

It is customary to express voltage and current in ac circuits by giving not the amplitude but $1/\sqrt{2}$ times the amplitude. This is often called the *root-mean-square* (rms) value: $V_{\text{rms}} = V_0/\sqrt{2}$. That takes care of the factor $1/2$ in Eq. (8.80), so that

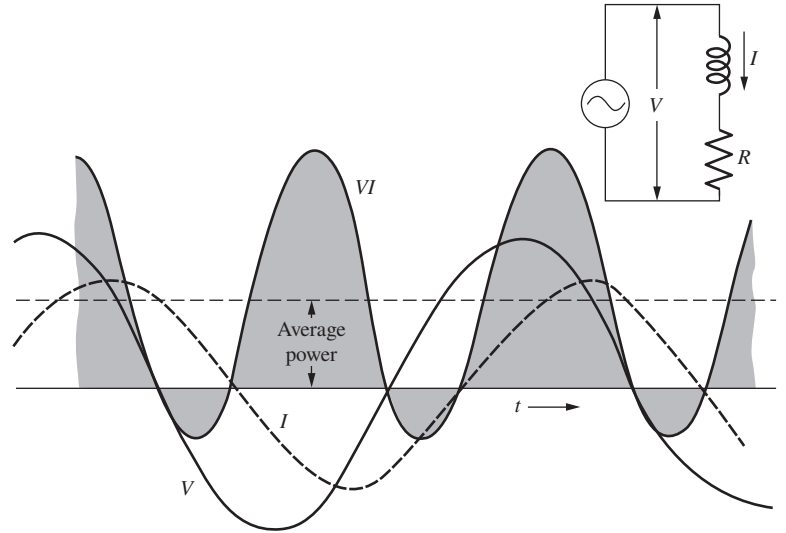
$$\boxed{\overline{P}_R = \frac{V_{\text{rms}}^2}{R}} \quad (8.81)$$

For example, the common domestic line voltage in North America is 120 volts, which corresponds to an *amplitude* $120\sqrt{2} = 170$ volts. The potential difference between the terminals of the electric outlet in your room (if the voltage is up to normal) is

$$V(t) = 170 \cos(377 \text{ s}^{-1} \cdot t), \quad (8.82)$$

where we have used the fact that the frequency is 60 Hz. An ac ammeter is calibrated to read 1 amp when the current amplitude is 1.414 amps.

Equation (8.81) holds in the case of a single resistor. More generally, the instantaneous rate at which energy is delivered to a circuit element (or a combination of circuit elements) is VI , the product of the total instantaneous voltage across the element(s) and the current, with due regard to sign. Consider this aspect of the current flow in the simple LR circuit in Fig. 8.5. In Fig. 8.23 we have redrawn the current and voltage graphs and added a curve proportional to the product VI . Positive VI means energy is being transferred into the LR combination from the source of electromotive force, or generator. Note that VI is negative in certain parts of the cycle. In those periods some energy is being returned to the generator. This is explained by the oscillation in the energy stored in the magnetic field of the inductor. This stored energy, $LI^2/2$, goes through a maximum twice in each full cycle.

**Figure 8.23.**

The instantaneous power VI is the rate at which energy is being transferred from the source of electromotive force on the left to the circuit elements on the right. The time average of this is indicated by the horizontal dashed line.

The *average power* $\bar{P}$ delivered to the LR circuit corresponds to the horizontal dashed line. To calculate its value, let's take a look at the product VI , with $V = \mathcal{E}_0 \cos \omega t$ and $I = I_0 \cos(\omega t + \phi)$:

$$\begin{aligned} VI &= \mathcal{E}_0 I_0 \cos \omega t \cos(\omega t + \phi) \\ &= \mathcal{E}_0 I_0 (\cos^2 \omega t \cos \phi - \cos \omega t \sin \omega t \sin \phi). \end{aligned} \quad (8.83)$$

The term proportional to $\cos \omega t \sin \omega t$ has a time average zero, as is obvious if you write it as $(1/2) \sin 2\omega t$, while the average of $\cos^2 \omega t$ is $1/2$. Thus for the time average we have

$$\bar{P} = \overline{VI} = \frac{1}{2} \mathcal{E}_0 I_0 \cos \phi. \quad (8.84)$$

If both current and voltage are expressed as rms values, in volts and amps, respectively, then

$$\boxed{\bar{P} = V_{\text{rms}} I_{\text{rms}} \cos \phi} \quad (8.85)$$

In this circuit all the energy dissipated goes into the resistance R . Naturally, any real inductor has some resistance. For the purpose of analyzing the circuit, we included that with the resistance R . Of course, the heat evolves at the actual site of the resistance.

The power P equals the product of the actual voltage $V(t)$ and actual current $I(t)$. These quantities in turn are the real parts of the complex voltage $\tilde{V}(t)$ and complex current $\tilde{I}(t)$. Does this mean that the power equals the real part of the product $\tilde{V}(t)\tilde{I}(t)$? Definitely not, because the product of the real parts doesn't equal the real part of the product; the real part of the product also has a contribution from the product of

the *imaginary* parts of $\tilde{V}(t)$ and $\tilde{I}(t)$. As we mentioned in Section 8.4, it doesn't make any sense to form the product of two complex quantities (excluding products with impedances and admittances, which are a different type of number; they aren't functions of time that we solve for). The point is that, since our original differential equations were linear in voltages and currents, we must keep things that way. The product of two of these quantities doesn't have anything to do with the actual solution to the differential equation.

There was nothing special about our *LR* circuit, so Eq. (8.85) holds for a general circuit (or subpart of a circuit), provided that V_{rms} is the total rms voltage across the circuit, I_{rms} is the rms current through the circuit, and ϕ is the phase between the instantaneous current and voltage. Equation (8.85) reduces to Eq. (8.81) in the special case where the circuit consists of a single resistor. In that case, the current across the resistor is in phase with the voltage, so $\phi = 0$. Additionally, $I_{\text{rms}} = V_{\text{rms}}/R$, so Eq. (8.85) simplifies to Eq. (8.81). In the case where a resistor is part of a larger circuit, remember that the V_{rms} in Eq. (8.85) is the voltage across the entire circuit (or whatever part we're concerned with), while the V_{rms} in Eq. (8.81) is the voltage across only the resistor; see Problem 8.14.

Example To get some more practice with the methods we developed in Section 8.5, we'll analyze the circuit in Fig. 8.24(a). A 10,000 ohm, 1 watt resistor (this rating gives the maximum power the resistor can safely absorb) has been connected up with two capacitors of capacitance 0.2 and 0.5 microfarads. We propose to plug this into the 120 volt, 60 Hz outlet. *Question:* Will the 1 watt resistor get too hot? In the course of finding out whether the average power dissipated in R exceeds the 1 watt rating, we'll calculate some of the currents and voltages we might expect to measure in this circuit. One way to work through the circuit is outlined below.

$$\text{Admittance of } C_2 = i\omega C_2 = i(377)(2 \cdot 10^{-7}) = 0.754 \cdot 10^{-4} i \text{ ohm}^{-1}$$

$$\text{Admittance of the resistor} = \frac{1}{R} = 10^{-4} \text{ ohm}^{-1}$$

$$\text{Admittance of } \begin{array}{c} \text{resistor} \\ \text{and } C_2 \end{array} = 10^{-4}(1 + 0.754i) \text{ ohm}^{-1}$$

$$\text{Impedance of } \begin{array}{c} \text{resistor} \\ \text{and } C_2 \end{array} = \frac{1}{10^{-4}(1 + 0.754i)} = \frac{10^4(1 - 0.754i)}{1^2 + 0.754^2} \\ = (6380 - 4810i) \text{ ohms}$$

$$\text{Impedance of } C_1 = -\frac{i}{\omega C} = -\frac{i}{(377)(5 \cdot 10^{-7})} = -5300i \text{ ohms}$$

$$\text{Impedance of entire circuit} = (6380 - 10,110i) \text{ ohms}$$

$$I_1 = \frac{120}{6380 - 10,110i} = \frac{120(6380 + 10,110i)}{(6380)^2 + (10,110)^2} = (5.36 + 8.49i) \cdot 10^{-3} \text{ amp}$$

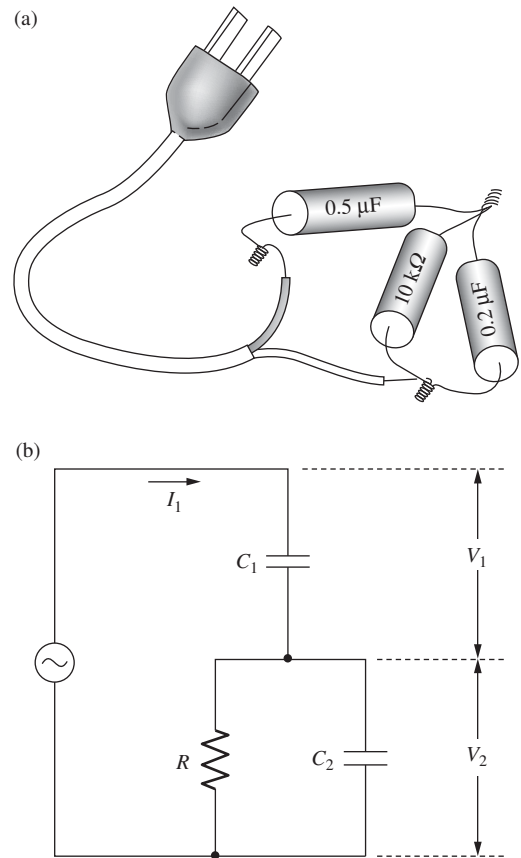


Figure 8.24.

An actual network (a) ready to be connected to a source of electromotive force, and (b) the circuit diagram.

Since 120 volts is the rms voltage, we obtain the rms current. That is, the modulus of the complex number I_1 , which is $[(5.36)^2 + (8.49)^2]^{1/2} \cdot 10^{-3}$ amp or 10.0 milliamps, is the rms current. An ac milliammeter inserted in series with the line would read 10 milliamps. This current has a phase angle $\phi = \tan^{-1}(0.849/0.536)$ or 1.01 radians with respect to the line voltage. From Eq. (8.85), the average power delivered to the entire circuit is then

$$\bar{P} = (120 \text{ volts})(0.010 \text{ amp}) \cos 1.01 = 0.64 \text{ watt.} \quad (8.86)$$

In this circuit the resistor is the only dissipative element, so this must be the average power dissipated in it. Just as a check, we can find the voltage V_2 across the resistor. If V_1 is the voltage across C_1 , we have

$$V_1 = I_1 \left(\frac{-i}{\omega C} \right) = (5.36 + 8.49i)(-5300i)10^{-3} = (45.0 - 28.4i) \text{ volts;}$$

$$V_2 = 120 - V_1 = (75.0 + 28.4i) \text{ volts.} \quad (8.87)$$

The current I_2 in R will be in phase with V_2 , of course, so the average power in R will be

$$\bar{P} = \frac{V_2^2}{R} = \frac{(75.0)^2 + (28.4)^2}{10^4} = 0.64 \text{ watt,} \quad (8.88)$$

which checks. Thus the rating of the resistor isn't exceeded, for what that assurance is worth. Actually, whether the resistor will get too hot depends not only on the average power dissipated in it, but also on how easily it can get rid of the heat. The power rating of a resistor is only a rough guide.

8.7 Applications

The *resonance* of electrical circuits has numerous applications in the modern world. Our lives wouldn't be the same without it. Any wire-less communication, from radios to cell phones to computers to GPS systems, is made possible by resonance. If you have a radio sitting on your desk, it is being bombarded by electromagnetic waves (discussed in Chapter 9) with all sorts of frequencies. If you want to pick out a particular frequency emitted by a radio station, you can "tune" your radio to that frequency by adjusting the radio's resonant frequency. This is normally done by adjusting the capacitance of the internal circuit by using *varactors* – diodes whose capacitance can be controlled by an applied voltage. Assuming that the resistance of the circuit is small, two things will happen when the resonant frequency matches the frequency of the radio station: there will be a large oscillation in the circuit at the radio station's frequency, and there will also be a negligible oscillation at all the other frequencies that are bombarding the radio. A high Q value of the circuit leads to both of these effects, due to the facts that the height of the peak in Fig. 8.11 is proportional to Q (as you can show) and that the width is proportional to $1/Q$. The oscillation in the circuit can then be demodulated (see the AM/FM discussion in Section 9.8) and amplified and sent to the speakers, creating the sound that you hear. Resonance

provides us with an astonishingly simple and automatic mechanism for finding needles in haystacks.

The microwaves in a *microwave oven* are created by a *magnetron*. This device consists of a ring-like chamber with a number (often eight) of cavities around the perimeter (Fig. 8.25). These cavities have both a capacitance and an inductance (and also a small resistance), so they act like little resonant *LC* circuits. Their size is chosen so that the resonant frequency is about 2.5 GHz. The charge on the tips of the little *LC* cavities alternates in sign around the perimeter of the ring. Charge (and hence energy) is added to the system by emitting electrons from the center of the ring. These electrons are attracted toward the positive tips. If this were the whole story, the effect would be to *reduce* the charge in the system. But there is a clever way of reversing the effect: by applying an appropriate magnetic field, the paths of the electrons can be bent by just the right amount to make them hit the *negative* tips. Charge is therefore added to the system instead of subtracted. The microwave radiation can be extracted by, say, inducing a current in small coils contained in the *LC* cavities.

The electricity that comes out of your wall socket is alternating current (ac) as opposed to direct current (dc). The rms voltage in North America is 120 V, and the frequency is 60 Hz. (In Europe the values are 230 V and 50 Hz, respectively.) The fundamental reason we use ac instead of dc is that, in the case of ac, it is easy to increase or decrease the voltage via a transformer. This is critical for the purpose of transmitting power over long distances, because for a given power $P = IV$ supplied by a power plant, a large V implies a small I , which in turn implies a small I^2R power loss in the long transmission lines. It is much more difficult to change the voltage in the case of dc. This was the deciding factor during the “War of Currents” in the 1880s, when ac and dc power were battling for dominance. Because dc power had to be shipped at the same low voltage at which it was used, dc power plants needed to be located within a few miles of the load. This had obvious disadvantages: cities would need to contain many power plants, and conversely a dam located far from a populated area would be useless. However, modern developments have made the conversion of dc voltages easier, so *high-voltage, direct current (HVDC)* power transmission is used in some instances. For both ac and dc, the long-haul voltages are on the order of a few hundred kilovolts. The War of Currents pitted (among many other people) Thomas Edison on the dc side against Nikola Tesla on the ac side.

Most of the electricity produced in power plants is *three-phase*. That is, there are three separate wires carrying voltages that are 120° out of phase. This can be achieved, for example, by having three loops of wire in Fig. 7.13 instead of just the one shown. There are various advantages to three-phase power, one of which is that it delivers a more steady power compared with single-phase, which has two moments during each cycle when the voltage is zero. However, this is mainly relevant for large

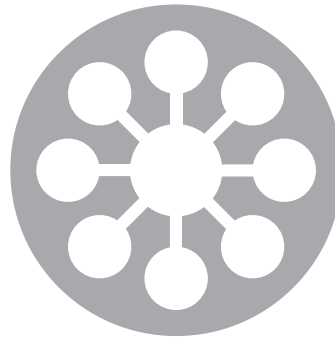


Figure 8.25.

A magnetron. The cavities have both a capacitance and an inductance.

machinery. Most households are connected to only one of the phases (or between two of them) in the power grid.

The ac power delivered to your home works fine for many electrical devices. For example, a toaster and an incandescent light bulb require only the generation of I^2R power, which is created by either ac or dc. But many other devices require dc, because the direction of the current in the electronic circuits matters. A *power adapter* converts ac to dc, while generally also lowering the voltage. The voltage is lowered by a transformer, and then the conversion to dc is accomplished by a *bridge rectifier*, which consists of a combination of four diodes that lets the current flow in only one direction. Additionally, a capacitor helps smooth out the dc voltage by storing charge and then releasing it when the voltage would otherwise dip.

As mentioned in [Section 3.9](#), it is advantageous to perform *power-factor correction* in the ac electrical power grid. The larger the imaginary part of an impedance of, say, an electrical motor, the larger the phase angle ϕ , and hence the smaller the $\cos \phi$ factor in Eq. (8.85), which is known as the *power factor*. At first glance, this doesn't seem to present a problem, because the unused power simply sloshes back and forth between the power station and the motor. However, for a given amount of net power consumed, a smaller power factor means that the current I will need to be larger. This in turn means that there will be larger I^2R power losses in the (generally long) transmission lines. For this reason, industries are usually charged a higher rate if their power factor is below 0.95. In an inductive circuit (for example, a motor with its many windings), the power factor can be increased by adding capacitance to the circuit, because this will reduce the magnitude of the imaginary part of the impedance.

CHAPTER SUMMARY

- The loop equation for a series RLC circuit (with no emf source) yields a linear differential equation involving three terms, one for each element. In the underdamped case, the solution for the voltage across the capacitor is

$$V(t) = e^{-\alpha t}(A \cos \omega t + B \sin \omega t), \quad (8.89)$$

where

$$\alpha = \frac{R}{2L} \quad \text{and} \quad \omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2}. \quad (8.90)$$

The solutions for the overdamped and critically damped cases take other forms. The *quality factor* of a circuit is given by

$$Q = \omega \cdot \frac{\text{energy stored}}{\text{average power dissipated}}. \quad (8.91)$$

- If we add to the series *RLC* circuit a sinusoidal emf source, $\mathcal{E}(t) = \mathcal{E}_0 \cos \omega t$, then the solution for the current is $I(t) = I_0 \cos(\omega t + \phi)$, where

$$I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \quad \text{and} \quad \tan \phi = \frac{1}{R\omega C} - \frac{\omega L}{R}. \quad (8.92)$$

This is the *steady-state* solution that survives after the *transient* solution from [Section 8.1](#) has decayed away. I_0 is maximum when ω equals the resonant frequency, $\omega_0 = 1/\sqrt{LC}$. The width of the $I_0(\omega)$ curve around the resonance peak is on the order of ω_0/Q .

- The series *RLC* circuit can also be solved by replacing the $\mathcal{E}_0 \cos \omega t$ term in the Kirchhoff differential equation with $\mathcal{E}_0 e^{i\omega t}$, and then guessing an *exponential solution* of the form $\tilde{I}(t) = \tilde{I} e^{i\omega t}$. The actual current $I(t)$ is obtained by taking the real part of $\tilde{I}(t)$.
- In alternating-current *networks*, currents and voltages can be represented by *complex numbers*. The real part of the complex number is the actual current or voltage. The complex current and voltage are related to each other via the complex *admittance* or *impedance*: $\tilde{I} = Y\tilde{V}$ or $\tilde{V} = Z\tilde{I}$. The admittances and impedances for the three circuit elements R, L, C are given in [Table 8.1](#). Admittances add in parallel, and impedances add in series.
- We have presented three different methods for solving alternating-current networks. See the summary at the end of [Section 8.5](#).
- The average *power* delivered to a circuit is

$$\bar{P} = \frac{1}{2} \mathcal{E}_0 I_0 \cos \phi = V_{\text{rms}} I_{\text{rms}} \cos \phi, \quad (8.93)$$

where the rms values are $1/\sqrt{2}$ times the peak values. This reduces to $\bar{P}_R = V_{\text{rms}}^2/R$ in the case of a single resistor.

Problems

8.1 Linear combinations of solutions *

Homogeneous linear differential equations have the property that the sum, or any linear combination, of two solutions is again a solution. (“Homogeneous” means there’s a zero on one side of the equation.) Consider, for example, the second-order equation (although the property holds for any order),

$$A\ddot{x} + B\dot{x} + Cx = 0. \quad (8.94)$$

Show that if $x_1(t)$ and $x_2(t)$ are solutions, then the sum $x_1(t) + x_2(t)$ is also a solution. Show that this property does *not* hold for the nonlinear differential equation $A\ddot{x} + B\dot{x}^2 + Cx = 0$.

8.2 Solving linear differential equations **

Consider the n th-order homogeneous linear differential equation

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = 0. \quad (8.95)$$

Show that the solutions take the form of $x(t) = A_i e^{r_i t}$, where the r_i depend on the a_j coefficients. *Hint:* If the (d/dt) derivatives were replaced by the letter z , then we would have an n th-order polynomial in z , which we know can be factored, by the fundamental theorem of algebra. (You can assume that the roots of this polynomial are distinct. Things are a little more complicated if there are double roots; this is discussed in the solution.)

8.3 Underdamped motion ***

A second-order homogeneous linear differential equation can be written in the general form of

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0, \quad (8.96)$$

where α and ω_0 are constants. (For the series *RLC* circuit in [Section 8.1](#), [Eq. \(8.2\)](#) gives these constants as $\alpha = R/2L$ and $\omega_0^2 = 1/LC$.) From [Problem 8.2](#) we know that there are two independent exponential solutions to this equation. Find these two solutions, and then show that, in the underdamped case where $\alpha < \omega_0$, the general solution can be written in the form of [Eq. \(8.10\)](#).

8.4 Overdamped RLC circuit **

Find the constants β_1 and β_2 in [Eq. \(8.15\)](#) by plugging an exponential trial solution into [Eq. \(8.2\)](#). If R is very large, what does the solution look like for large t ?

8.5 Change in frequency **

For the decaying signal shown in [Exercise 8.19](#), estimate the percentage by which the frequency differs from the natural frequency $1/\sqrt{LC}$ of the circuit.

8.6 Limits of an RLC circuit ***

- In the $R \rightarrow 0$ limit, verify that the solution in [Eq. \(8.4\)](#) correctly reduces to the solution for an *LC* circuit. That is, show that the voltage behaves like $\cos \omega_0 t$.
- In the $L \rightarrow 0$ limit, verify that the solution in [Eq. \(8.15\)](#) correctly reduces to the solution for an *RC* circuit. That is, show that the voltage behaves like $e^{-t/RC}$. You will need to use the results from [Problem 8.4](#).

- (c) In the $C \rightarrow \infty$ limit, verify that the solution in Eq. (8.15) correctly reduces to the solution for an RL circuit. That is, show that the voltage behaves like $e^{-(R/L)t}$, up to an additive constant. What is the physical meaning of this constant?

8.7 *Magnitude and phase* *

Show that $a + bi$ can be written as $I_0 e^{i\phi}$, where $I_0 = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}(b/a)$.

8.8 *RLC circuit via vectors* ***

- (a) The loop equation for the series RLC in Fig. 8.26 is

$$L \frac{dI}{dt} + RI + \frac{Q}{C} = \mathcal{E}_0 \cos \omega t, \quad (8.97)$$

where we have taken positive I to be clockwise and Q to be the charge on the right plate of the capacitor. If I takes the form of $I(t) = I_0 \cos(\omega t + \phi)$, show that Eq. (8.97) can be written as

$$\begin{aligned} \omega L I_0 \cos(\omega t + \phi + \pi/2) + R I_0 \cos(\omega t + \phi) \\ + \frac{I_0}{\omega C} \cos(\omega t + \phi - \pi/2) = \mathcal{E}_0 \cos \omega t. \end{aligned} \quad (8.98)$$

- (b) At any given time, the four terms in Eq. (8.98) can be considered to be the real parts of four vectors in the complex plane. Draw the appropriate quadrilateral that represents the fact that the sum of the three terms on the left side of the equation equals the term on the right side.
- (c) Use your quadrilateral to determine the amplitude I_0 and phase ϕ of the current, and check that they agree with the values in Eqs. (8.38) and (8.39).

8.9 *Drawing the complex vectors* **

For the series and parallel RLC circuits in Figs. 8.10 and 8.20, draw the vectors representing all of the complex voltages and currents. For the sake of making a concrete picture, assume that $R = |Z_L| = 2|Z_C|$. The vectors all rotate around in the complex plane, so you can draw them at whatever instant in time you find most convenient.

8.10 *Real impedance* *

Is it possible to find a frequency at which the impedance at the terminals of the circuit in Fig. 8.27 will be purely real?

8.11 *Light bulb* *

A 120 volt (rms), 60 Hz line provides power to a 40 watt light bulb. By what factor will the brightness decrease if a $10 \mu\text{F}$ capacitor is connected in series with the light bulb? (Assume that the brightness is proportional to the power dissipated in the bulb's resistor.)

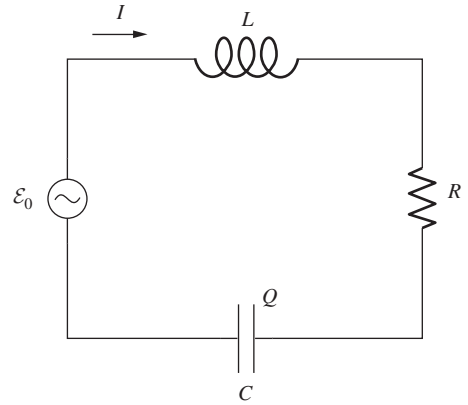


Figure 8.26.

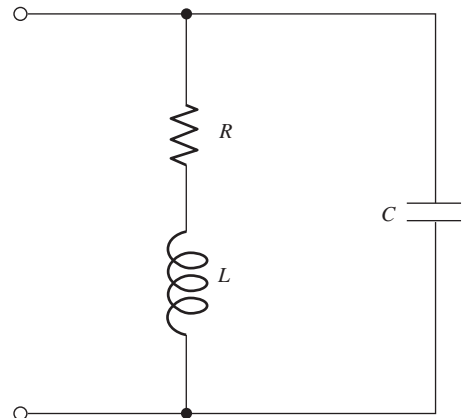


Figure 8.27.

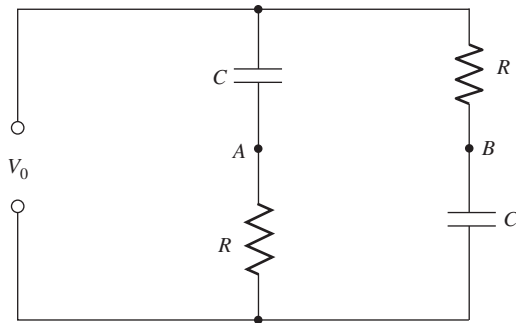


Figure 8.28.

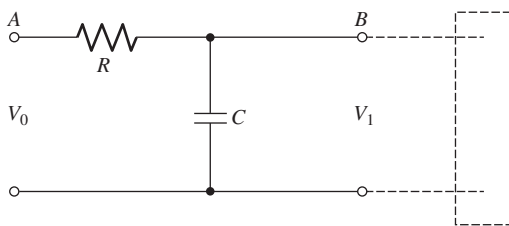


Figure 8.29.

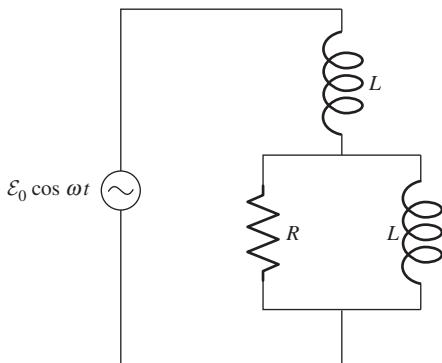


Figure 8.30.

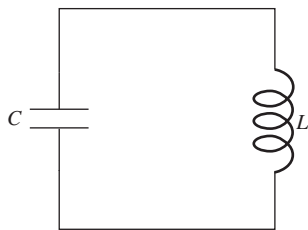


Figure 8.31.

8.12 Fixed voltage magnitude **

Let $V_{AB} \equiv V_B - V_A$ in the circuit in Fig. 8.28. Show that $|V_{AB}|^2 = V_0^2$ for any frequency ω . Find the frequency for which V_{AB} is 90° out of phase with V_0 .

8.13 Low-pass filter **

In Fig. 8.29 an alternating voltage $V_0 \cos \omega t$ is applied to the terminals at A. The terminals at B are connected to an audio amplifier of very high input impedance. (That is, current flow into the amplifier is negligible.) Calculate the ratio $|\tilde{V}_1|^2/V_0^2$. Here $|\tilde{V}_1|$ is the absolute value of the complex voltage amplitude at terminals B. Choose values for R and C to make $|\tilde{V}_1|^2/V_0^2 = 0.1$ for a 5000 Hz signal. This circuit is the most primitive of “low-pass” filters, providing attenuation that increases with increasing frequency. Show that, for sufficiently high frequencies, the signal power is reduced by a factor $1/4$ for every doubling of the frequency. Can you devise a filter with a more drastic cutoff – such as a factor $1/16$ per octave?

8.14 Series RLC power **

Consider the series RLC circuit in Fig. 8.10. Show that the average power delivered to the circuit, which is given in Eq. (8.84), equals the average power dissipated in the resistor, which is given in Eq. (8.80). (These equations are a little easier to work with than the equivalent rms equations, Eqs. (8.85) and (8.81).)

8.15 Two inductors and a resistor **

The circuit in Fig. 8.30 has two equal inductors L and a resistor R . The frequency of the emf source, $\mathcal{E}_0 \cos \omega t$, is chosen to be $\omega = R/L$.

- What is the total complex impedance of the circuit? Give it in terms of R only.
- If the total current through the circuit is written as $I_0 \cos(\omega t + \phi)$, what are I_0 and ϕ ?
- What is the average power dissipated in the circuit?

Exercises

8.16 Voltages and energies *

Consider the LC circuit in Fig. 8.31. Initial conditions have been set up so that the voltage change across the capacitor (proceeding around the loop in a clockwise manner) equals $V_0 \cos \omega t$, where $\omega = 1/\sqrt{LC}$. At $t = 0$, what are the voltage changes (proceeding clockwise) across the capacitor and inductor? Where is the energy stored? Answer the same questions for $t = \pi/2\omega$.

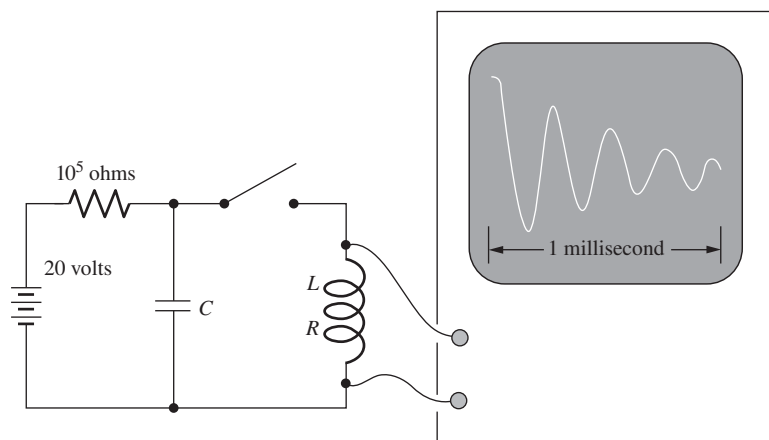


Figure 8.32.

8.17 *Amplitude after Q cycles **

In the RLC circuit in Section 8.1, show that the current (or voltage) amplitude decreases by a factor of $e^{-\pi} \approx 0.043$ after Q cycles.

8.18 *Effect of damping on frequency ***

Using Eqs. (8.9) and (8.13), express the effect of damping on the frequency of a series RLC circuit, by writing ω in terms of Q and $\omega_0 = 1/\sqrt{LC}$. Suppose enough resistance is added to bring Q from ∞ down to 1000. By what percentage is the frequency ω thereby shifted from ω_0 ? How about if Q is brought from ∞ down to 5?

8.19 *Decaying signal ***

The coil in the circuit shown in Fig. 8.32 is known to have an inductance of 0.01 henry. When the switch is closed, the oscilloscope sweep is triggered. The 10^5 ohm resistor is large enough (as you will discover) so that it can be treated as essentially infinite for parts (a) and (b) of this problem.

- Determine as well as you can the value of the capacitance C .
- Estimate the value of the resistance R of the coil.
- What is the magnitude of the voltage across the oscilloscope input a long time, say 1 second, after the switch has been closed?

8.20 *Resonant cavity ***

A resonant cavity of the form illustrated in Fig. 8.33 is an essential part of many microwave oscillators. It can be regarded as a simple LC circuit. The inductance is that of a rectangular toroid with one turn; see Eq. (7.62). This inductor is connected directly to a parallel-plate capacitor. Find an expression for the resonant frequency of this circuit, and show by a rough sketch the configuration of the magnetic and electric fields.

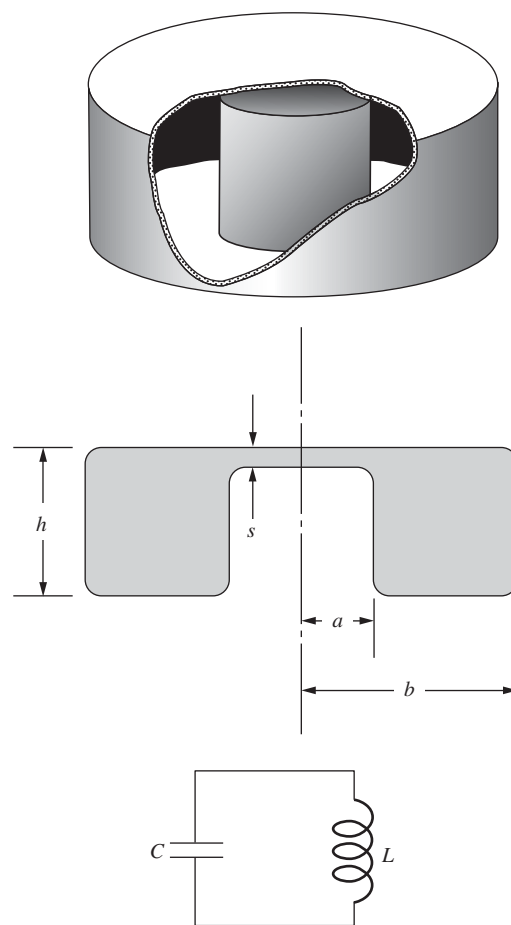


Figure 8.33.

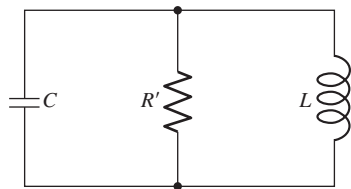


Figure 8.34.

8.21 *Solving an RLC circuit* ***

In the resonant circuit in Fig. 8.34 the dissipative element is a resistor R' connected in parallel, rather than in series, with the LC combination. Work out the equation, analogous to Eq. (8.2), that applies to this circuit. Find also the conditions on the solution analogous to those that hold in the series RLC circuit. If a series RLC and a parallel $R'LC$ circuit have the same L , C , and Q (quality factor, not charge), how must R' be related to R ?

8.22 *Overdamped oscillator* **

For the circuit in Fig. 8.4(a), determine the values of β_1 and β_2 for the overdamped case, with $R = 600$ ohms. Determine also the ratio of B to A , the constants in Eq. (8.15). You can use the results from Problem 8.4.

8.23 *Energy in an RLC circuit* ***

For the damped RLC circuit of Fig. 8.2, work out an expression for the total energy stored in the circuit (the energy in the capacitor plus the energy in the inductor) at any time t , for all three of the underdamped, overdamped, and critically damped cases; you need not simplify your answers. If R is varied while L and C are kept fixed, show that the critical damping condition, $R = 2\sqrt{L/C}$, is the one in which the total energy is most quickly dissipated. (The exponential behavior is all that matters here.) The results from Problem 8.4 will be useful.

8.24 *RC circuit with a voltage source* **

A voltage source $\mathcal{E}_0 \cos \omega t$ is connected in series with a resistor R and a capacitor C . Write down the differential equation expressing Kirchhoff's law. Then guess an exponential form for the current, and take the real part of your solution to find the actual current. Determine how the amplitude and phase of the current behave for very large and very small ω , and explain the results physically.

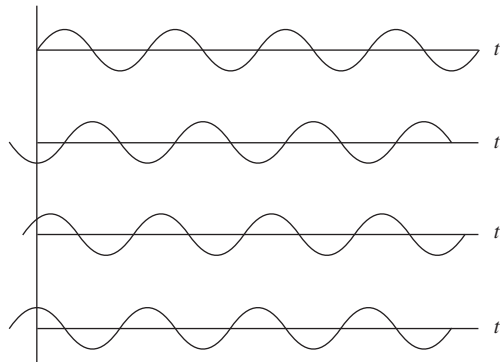
 $R, L, C, \mathcal{E}$ 

Figure 8.35.

8.25 *Light bulb* **

How large an inductance should be connected in series with a 120 volt (rms), 60 watt light bulb if it is to operate normally when the combination is connected across a 240 volt, 60 Hz line? (First determine the inductive reactance required. You may neglect the resistance of the inductor and the inductance of the light bulb.)

8.26 *Label the curves* **

The four curves in Fig. 8.35 are plots, in some order, of the applied voltage and the voltages across the resistor, inductor, and capacitor of a series RLC circuit. Which is which? Whose impedance is larger, the inductor's or the capacitor's?

8.27 *RLC parallel circuit* **

A 1000 ohm resistor, a 500 picofarad capacitor, and a 2 millihenry inductor are connected in parallel. What is the impedance of this combination at a frequency of 10 kilocycles per second? At a frequency of 10 megacycles per second? What is the frequency at which the absolute value of the impedance is greatest?

8.28 *Small impedance* *

Consider the circuit in Fig. 8.36. The frequency is chosen to be $\omega = 1/\sqrt{LC}$. Given L and C , how should you pick R so that the impedance of the circuit is small?

8.29 *Real impedance* *

Is it possible to find a frequency at which the impedance at the terminals of the circuit in Fig. 8.37 will be purely real?

8.30 *Equal impedance?* *

Do there exist values of R , L , and C for which the two circuits in Fig. 8.38 have the same impedance? (The resistor R has the same value in both.) Can you give a physical explanation why or why not?

8.31 *Zero voltage difference* **

Show that, if the condition $R_1 R_2 = L/C$ is satisfied by the components of the circuit in Fig. 8.39, the difference in voltage between points A and B will be zero at any frequency. Discuss the suitability of this circuit as an ac bridge for measurement of an unknown inductance.

8.32 *Finding L* **

In the laboratory you find an inductor of unknown inductance L and unknown internal resistance R . Using a dc ohmmeter, an ac voltmeter of high impedance, a 1 microfarad capacitor, and a 1000 Hz signal generator, determine L and R as follows. According to the ohmmeter, R is 35 ohms. You connect the capacitor in series with the inductor and the signal generator. The voltage across both is 10.1 volts. The voltage across the capacitor alone is 15.5 volts. You note also, as a check, that the voltage across the inductor alone is 25.4 volts. How large is L ? Is the check consistent?

8.33 *Equivalent boxes* ***

Show that the impedance Z at the terminals of each of the two circuits in Fig. 8.40 is (ignoring the units)

$$Z = \frac{5000 + 16 \cdot 10^{-3} \omega^2 - 16i\omega}{1 + 16 \cdot 10^{-6} \omega^2}. \quad (8.99)$$

Since they present, at any frequency, the identical impedance, the two black boxes are completely equivalent and indistinguishable

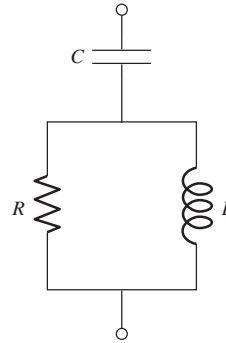


Figure 8.36.

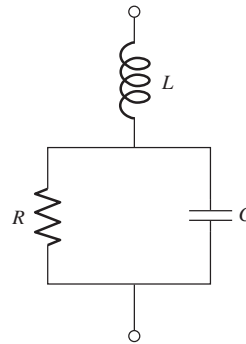


Figure 8.37.

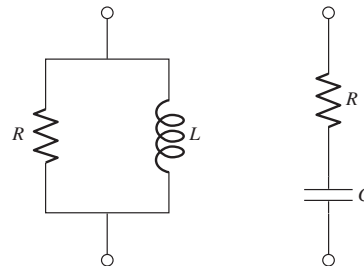


Figure 8.38.

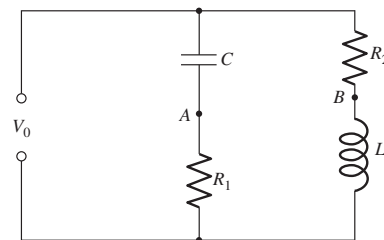


Figure 8.39.

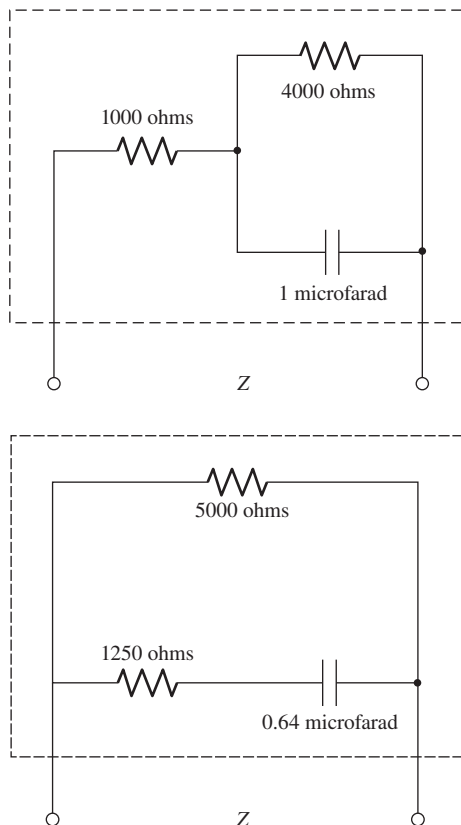


Figure 8.40.

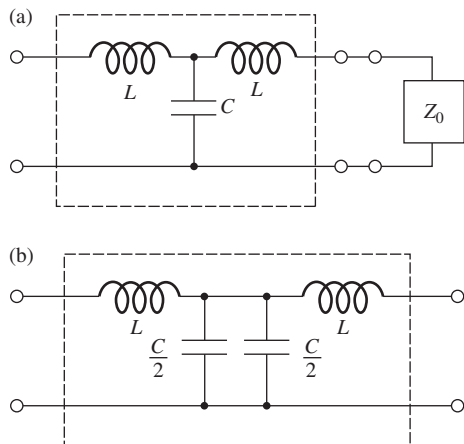


Figure 8.41.

from the outside. See if you can discover the general rules for finding the resistances and capacitance in the bottom box, given the resistances and capacitance in the top box.

8.34 *LC chain* **

The box in Fig. 8.41(a) with four terminals contains a capacitor C and two inductors of equal inductance L connected as shown. An impedance Z_0 is to be connected to the terminals on the right. For given frequency ω , find the value that Z_0 must have if the resulting impedance between the terminals on the left (the “input” impedance) is to be equal to Z_0 .

(You will find that the required value of Z_0 is a pure resistance R_0 provided that $\omega^2 < 2/LC$. A chain of such boxes could be connected together to form a ladder network resembling the ladder of resistors in Exercise 4.36. If the chain is terminated with a resistor of the correct value R_0 , its input impedance at frequency ω will be R_0 , no matter how many boxes make up the chain.)

What is Z_0 in the special case $\omega = \sqrt{2/LC}$? It helps in understanding that case to note that the contents of the box (a) can be equally well represented by box (b).

8.35 *RC circuit* **

A $2000\ \text{ohm}$ resistor and a $1\ \text{microfarad}$ capacitor are connected in series across a $120\ \text{volt (rms)}$, $60\ \text{Hz}$ line.

- What is the total impedance?
- What is the rms value of the current?
- What is the average power dissipated in the circuit?
- What will be the reading of an ac voltmeter connected across the resistor? Across the capacitor?
- The left and right plates of a cathode ray tube are connected across the resistor, and the top and bottom plates are connected across the capacitor. The horizontal and vertical axes of the tube’s screen therefore indicate the voltages across the resistor and capacitor, respectively. Sketch the pattern that you expect to see on the screen. From the given information, is it possible to determine the direction in which the pattern is traced out?

8.36 *High-pass filter* **

Consider the setup in Problem 8.13, but with the capacitor replaced by an inductor. Calculate the ratio $|\tilde{V}_1|^2/V_0^2$. Choose values for R and L to make $|\tilde{V}_1|^2/V_0^2 = 0.1$ for a $100\ \text{Hz}$ signal. This circuit is the most primitive of “high-pass” filters, providing attenuation that increases with decreasing frequency. Show that, for sufficiently low frequencies, the signal power is reduced by a factor $1/4$ for every halving of the frequency.

8.37 *Parallel RLC power* **

Repeat the task of Problem 8.14, but now for the parallel *RLC* circuit in Fig. 8.20.

8.38 *Two resistors and a capacitor* **

The circuit in Fig. 8.42 has two equal resistors R and a capacitor C . The frequency of the emf source, $\mathcal{E}_0 \cos \omega t$, is chosen to be $\omega = 1/RC$.

- (a) What is the total complex impedance of the circuit? Give it in terms of R only.
- (b) If the total current through the circuit is written as $I_0 \cos(\omega t + \phi)$, what are I_0 and ϕ ?
- (c) What is the average power dissipated in the circuit?

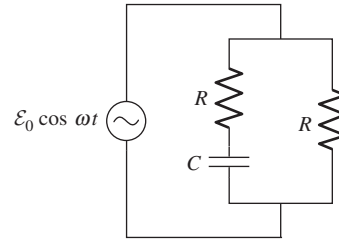


Figure 8.42.